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Two examples of functional penalisations of Brownian motion, VIII

B. Roynette[(1)], M. Yor[(2), (3)]

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[(1)] Institut Elie Cartan, Université Henri Poincaré,
B.P. 239, 54506 Vandoeuvre les Nancy Cedex

[(2)] Laboratoire de Probabilités et Modèles Aléatoires,
Université Paris VI et VII, 4 place Jussieu - Case 188
F - 75252 Paris Cedex 05

[(3)] Institut Universitaire de France

Abstract On one hand, we penalise Brownian paths by a function of the one-sided supremum of Brownian motion, considered up to the last, resp. the first, zero, before t, resp. after t. This study provides some analogy with penalisations by the longest length of Brownian excursions, up to time t.
On the other hand, we describe Feynman-Kac type penalisation results for long Brownian bridges, thus completing some similar study for free Brownian motion.

Key words Penalisation of Brownian paths, one-sided supremum, Feynman-Kac functionals, long Brownian bridges.

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1 Introduction.

1.1 Notations.

Throughout this work, $\big(\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty = \underset{t\geq 0}{\vee} \mathcal{F}_t, P_x(x \in \mathbb{R})\big)$ denotes the canonical one-dimensional Wiener process. $\Omega = \mathcal{C}\big([0,\infty[\to \mathbb{R}\big)$ is the space of continuous functions $(X_t, t \geq 0)$ the process of coordinates on this space, $(\mathcal{F}_t, t \geq 0)$ its natural filtration and $(P_x, x \in \mathbb{R})$ the family of Wiener measures on $(\Omega, \mathcal{F}_\infty)$, with $P_x(X_0 = x) = 1$. When $x = 0$, we write simply P for P_0.

For every $t \geq 0$, let $g_t := \sup\{s \leq t, X_s = 0\}$ denote the last zero before t and $d_t := \inf\{s \geq t, X_s = 0\}$ the first zero after t. Thus $d_t - g_t$ is the duration of the excursion which straddles t.

For every $t \geq 0$, let $S_t := \sup\limits_{s\leq t} X_s$. The increasing process $(S_t, t \geq 0)$ is the process of the one-sided supremum of X. We also denote : $X_t^* = \sup\limits_{s\leq t} |X_s|$.

For every $a \in \mathbb{R}$, $T_a := \inf\{t,\, X_t = a\}$ denotes the first hitting time of level a by the process $(X_t,\, t \geq 0)$.

We denote by $(L_t,\, t \geq 0)$ the (continuous) local time process at level 0 for $(X_t,\, t \geq 0)$ and by $(\tau_l,\, l \geq 0)$ its right-continuous inverse :

$$\tau_l := \inf\{s \,;\, L_s > l\}, \qquad l \geq 0$$

For every $t \geq 0$, we denote by θ_t the operator of translation in time of the Brownian trajectory : $X_s \circ \theta_t = X_{s+t} \quad (s,t \geq 0)$.

Finally, if a is a real number, $a^+ = \sup(0,a)$ and $a^- = -\inf(0,a)$.

This paper consists of two distinct parts A and B which may be read independently from each other.

1.2 Introduction to part A : penalisations by $\varphi(S_{g_t}), \varphi(S_{d_t})$ and $\varphi(X^*_{g_t})$.

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a probability density. φ is a Borel function with integral equal to 1. We denote by Φ the primitive of φ which is equal to 0 at $x = 0$.

$$\Phi(x) := \int_0^x \varphi(y)dy \qquad (x \geq 0)$$

1.2.1 In [RVY, II], we studied the penalisation by $\varphi(S_t)$.

More precisely, we have shown that :

- for every $s \geq 0$, $\Lambda_s \in \mathcal{F}_s$:

$$\lim_{t\to\infty} \frac{E\big(1_{\Lambda_s}\varphi(S_t)\big)}{E\big(\varphi(S_t)\big)} = E(1_{\Lambda_s}\widetilde{M}^\varphi_s) := \widetilde{Q}^\varphi(\Lambda_s) \tag{1.1}$$

  with :

$$\widetilde{M}^\varphi_s := \varphi(S_s)(S_s - X_s) + \big(1 - \Phi(S_s)\big) \tag{1.2}$$

  and $(\widetilde{M}^\varphi_s,\, s \geq 0)$ is a $(\mathcal{F}_s,\, s \geq 0,\, P)$ martingale,

- then, we described precisely the canonical process $\big(\Omega,\, (X_t,\, t \geq 0),\, \mathcal{F}_\infty\big)$ under the probability $\widetilde{Q}^\varphi$ induced by (1.1).

1.2.2 Let, for every $t \geq 0$, $V^{(1)}_{g_t} := \sup\{d_s - g_s \,;\, d_s \leq t\}$ denote the length of the longest excursion before g_t. Let $A_t := t - g_t$; the process $(A_t,\, t \geq 0)$ is the process of the age (of excursions) ; we denote $A^*_t = \sup_{s\leq t} A_t$. Finally, let for every $t \geq 0$:

$$V^{(1)}_{d_t} := \sup_{s\leq t}\{d_s - g_s \,;\, g_s \leq\; t\} = V^{(1)}_{g_t} \vee (d_t - g_t)$$

denote the length of the longest excursion before d_t. Of course, the following holds :

$$V^{(1)}_{g_t} \leq A^*_t \leq V^{(1)}_{d_t}. \tag{1.3}$$

In [RVY, VII], we have studied, in particular, the penalisation by $\varphi(V^{(1)}_{g_t})$ $\big($resp. $\varphi(A^*_t)$, resp. $\varphi(V^{(1)}_{d_t})\big)$. In other terms, we studied the penalisation by a function of the length of the longest excursion before g_t (resp. before t, resp. before d_t).

1.2.3 Comparing the above points 1.2.1 and 1.2.2, it is now natural to study

the penalisation by a function of the highest (and not the longest)(positive) excursion before g_t (resp. before t, resp. before d_t). The study before t has been derived in [RVY, II], as we recalled in 1.2.1 above. The aim of part A of the present work is to study the penalisation of Wiener measure by the functionals $\varphi(S_{g_t})$ and $\varphi(S_{d_t})$, i.e. : by a function of the highest height of positive excursions before g_t and before d_t. In the same spirit, we shall study the penalisation by $\varphi(X^*_{g_t})$, with

$$X^*_{g_t} := \sup_{s \leq g_t} |X_s| \tag{1.4}$$

1.3 Introduction to part B.

This part B completes the paper [RVY, I].

Let $q : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a Borel function, and $A^q_t := \displaystyle\int_0^t q(X_s)ds$; as previously, $(X_t,\, t \geq 0)$ denotes a one-dimensional Brownian motion. In [RVY, I], we show, under some adequate hypotheses, that :

$$E_x\left(\exp - \frac{1}{2}A^q_t\right) \underset{t\to\infty}{\sim} \frac{\varphi_q(x)}{\sqrt{t}} \tag{1.5}$$

where φ_q is a function depending upon q in a simple manner, and we study the penalisation of Wiener measures by the functional $\Gamma_t := \exp\Big(- \frac{1}{2}A^q_t\Big)$. More explicitly, we show that for every $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\lim_{t\to\infty} \frac{E_x\Big(1_{\Lambda_s}\exp - \dfrac{1}{2}A^q_t\Big)}{E_x\Big(\exp - \dfrac{1}{2}A^q_t\Big)} = E_x(1_{\Lambda_s} M^q_s) \tag{1.6}$$

with

$$M^q_s := \frac{\varphi_q(X_s)}{\varphi_q(x)} \exp\Big(- \frac{1}{2}A^q_s\Big)$$

Here, we shall complete these results in 3 directions.

<u>In part B.1</u>, we use the results of [RVY, I] to prove a local limit Theorem concerning the Brownian additive functional A^q_t. More precisely, we prove that, for every real x, there exists a positive measure ν_x, (depending on q) carried by $\mathbb{R}_+$ and $\sigma-$finite such that :

$$\sqrt{t}\, P_x(A^q_t \in dz) \underset{t\to\infty}{\longrightarrow} \nu_x(dz) \qquad \text{(see Theorem B.1)} \tag{1.7}$$

and we give the explicit value of ν_x for many examples of functions q.

<u>In part B.2</u>, we take for q the function $q_0(x) = 1_{]-\infty,0]}(x)$, hence :

$$A^{q_0}_t := A^-_t = \int_0^t 1_{(X_s<0)}ds.$$

Now, let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that : $\displaystyle\int_0^\infty \frac{h(a)da}{\sqrt{a}} < \infty$. The explicit knowledge of the law of A^-_t under P_x, for every real x, allows us to study in detail the following limit :

$$\frac{E_x\big(1_{\Lambda_s}h(A^-_t)\big)}{E_x\big(h(A^-_t)\big)} \underset{t\to\infty}{\longrightarrow} E_x(1_{\Lambda_s}M^{-,h}_s) := Q^{-,h}(\Lambda_s) \tag{1.8}$$

where the martingale $(M_s^{-,h},\, s \geq 0)$ is known explicitly. Then, we study in a detailed manner the canonical process $(X_t,\, t \geq 0)$ under $Q^{-,h}$ (see Theorem B 6 below, where we also consider the penalisation with "long Brownian bridges").

In part B.3, the convergence results (1.5) and (1.8) are extended to the "long Brownian bridges". In particular,we show (see Theorem B.10) :

$$t\, E_x\Big(\exp - \frac{1}{2} A_t^q | X_t = y\Big) \underset{t\to\infty}{\longrightarrow} \frac{\pi}{2} \varphi_q(x) \varphi_q(y)$$

and

$$t\, P_x(A_t^q \in dz | X_t = y) \underset{t\to\infty}{\longrightarrow} \nu_x * \nu_y(dz)$$

where ν_x (resp. ν_y) is defined via (1.7).

1.4 The relative position of this paper in our penalisaton studies.

Since roughly 2002, we have devoted most of our research activities to various kinds of penalisations of Brownian paths ; two sets of papers are emerging from these studies : essentially, the first set, with Roman numberings, going from I to VIII, of which the present paper is the last item, discusses "individual" cases of penalisations, whereas the second set consists in a monograph [RY ; M] made up of 5 Chapters ; let us now discuss a little more in detail these two sets : a) "The Roman set" ; b) The Monograph [RY,M].

• The "Roman set" a) consists in a number of detailed studies of penalisation of Brownian paths with various functionals, including :

- continuous additive functionals such as $A_t^q = \displaystyle\int_0^t X_s ds$ [RVY, I], (we now call these Feynman-Kac penalisations) ;
- the one-sided suprenum : $S_t = \sup\limits_{s\leq t} X_s$; or the local time at 0 $\big($[RVY, II]$\big)$;
- lengths of excursions, ranked in decreasing order [RVY, VII].

This latter study led us, at no big extra cost, to work in the set-up of d-dimensional Bessel processes, for $0 < d < 2$, since the Brownian arguments may be extended there in a natural manner $\big($[RVY, V] [RVY, VII]$\big)$.

We also developed penalisation studies in the context of planar brownian and its winding process $\big($[RVY, VI]$\big)$. The present paper complements [RVY, I and II].

• In the "Monograph" b), we attempt to develop a global viewpoint about penalisation, e.g. concerning the Feynman-Kac type penalisations, we exhibit some σ-finite measures on path space which "rule" jointly all these penalisations. See also J. Najnudel's thesis [N], which gives some full proofs to certain "meta-theorems" presented in [RY,M].

The same goes for the lengths of excursions penalisation studies. Finally, although the present paper is closely connected with our previous studies, it has been written, and may be read, independently of the contents of a) and b), and we hope that, despite its numbering as the last of the list of a), it may be used as an introduction to both a) and b).

2 Part A. 1. Penalisations by $\varphi(S_{g_t}), \varphi(S_{d_t})$ and $\varphi(X^*_{g_t})$

A.1. Penalisation by $\varphi(S_{g_t})$.
Recall that $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a probability density on $\mathbb{R}_+$:

$$\int_0^\infty \varphi(y)dy = 1 \tag{2.1}$$

$$\Phi(x) := \int_0^x \varphi(y)dy. \tag{2.2}$$

Theorem A.1. *Under the preceding hypothesis :*

1) *For every $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} \frac{E\big(1_{\Lambda_s}\varphi(S_{g_t})\big)}{E\big(\varphi(S_{g_t})\big)} = E[1_{\Lambda_s} M_s^\varphi] \tag{2.3}$$

with :

$$M_s^\varphi := \frac{1}{2}\varphi(S_{g_s})|X_s| + \varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s) \tag{2.4}$$

Moreover, $(M_s^\varphi,\, s \geq 0)$ is a positive martingale, which writes :

$$M_s^\varphi = 1 + \frac{1}{2}\int_0^s \varphi(S_{g_u}) sgn\, X_u\, dX_u - \int_0^s \varphi(S_u) 1_{X_u>0} dX_u. \tag{2.5}$$

2) *The formula :*

$$Q^\varphi[\Lambda_s] := E(1_{\Lambda_s} M_s^\varphi) \quad (\Lambda_s \in \mathcal{F}_s) \tag{2.6}$$

induces a probability Q^φ on $(\Omega, \mathcal{F}_\infty)$. Under Q^φ, the canonical process $(X_t,\, t \geq 0)$ satisfies the following :

i) let $g := \sup\{t\ ;\ X_t = 0\}$. Then :

$$Q^\varphi\{0 < g < \infty\} = 1 \tag{2.7}$$

ii) the couple $(L_g, S_g) \equiv (L_\infty, S_g)$ admits the density :

$$f^{Q^\varphi}_{L_g,S_g}(v,c) = \frac{1}{4}\frac{v}{c^2}\, e^{-\frac{v}{2c}} \varphi(c) 1_{v>0,c>0} \tag{2.8}$$

In particular, S_g admits φ as a density, $\dfrac{1}{2}\dfrac{L_g}{S_g}$ is a gamma variable, with parameter 2 and S_g and $\dfrac{L_g}{S_g}$ are independent ;

iii) $Q^\varphi\{S_\infty = \infty\} = \dfrac{1}{2}$ and, conditionally on $S_\infty < \infty$, S_∞ admits φ as its density.

3) *Under Q^φ :*

i) $(X_t,\, t \leq g)$ and $(X_t,\, t \geq g)$ are two independent processes ;

ii) with probability $\frac{1}{2}$, $(X_{g+t},\ t \geq 0)$ $\big($*resp.* $(-X_{g+t},\, t \geq 0)\big)$ *is a Bessel process with dimension 3 ;*

iii) conditionally upon $L_g = v$ *and* $S_g = c$, *the process* $(X_t,\, t \leq g)$ *is a Brownian motion stopped at* τ_v *and conditioned upon* $S_{\tau_v} = c$.

4) *Under* Q^φ, $\big(|X_t| + L_t,\, t \geq 0\big)$ *is a* 3*-dimensional Bessel, independent from* (S_g, L_g)

Proof of Theorem A.1.

A.1.1. We shall begin to gather, and give simple proofs of, some classical results relative to the laws of certain Brownian r.v.'s.

Proposition A.2. *Under Wiener measure* P *:*

$$\mathbf{1})\quad \textit{For any } t \geq 0\ :\ S_{g_t} \overset{(law)}{=} \frac{1}{2}\sqrt{t}|N|, \quad \textit{with } N \textit{ a reduced Gaussian r.v.} \tag{2.9}$$

$$\mathbf{2})\quad \textit{For any } a > 0,\ S_{g_{T_a}} \textit{ is uniform on } [0,a]. \tag{2.10}$$

$$\mathbf{3})\quad \textit{For any } l > 0,\ S_{\tau_l} \textit{ admits as density } : f_{S_{\tau_l}}(c) = \frac{l}{2c^2}\, e^{-\frac{l}{2c}} 1_{c \geq 0}. \tag{2.11}$$

$$\mathbf{4})\quad \textit{Let } a < 0.\ S_{T_a} \textit{ admits as density } : f_{S_{T_a}}(c) = -\frac{a}{(c-a)^2}\, 1_{c>0}. \tag{2.12}$$

$$\textit{i.e. } S_{T_a} \overset{(law)}{=} a - \frac{a}{U} \textit{ where } U \textit{ is uniform as } [0,1].$$

$$\mathbf{5})\quad \textit{Let } a > 0. \textit{ Under } P_a,\ S_{T_0} \overset{(law)}{=} \frac{a}{U}. \tag{2.13}$$

where U *is uniform sur* $[0,1]$.

From this point 5, we deduce that, for every integrable function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$:

$$E\big[\psi(S_{d_t})|\mathcal{F}_t\big] = \widehat{E}\Big[\psi\Big(S_t \vee \frac{X_t^+}{\widehat{U}}\Big)\Big] \tag{2.14}$$

where $\widehat{U}$ is uniform on $[0,1]$ and independent from $\mathcal{F}_t$. In this expression (2.14) the letters without hats are frozen, and the expectation is taken uniquely with respect to $\widehat{U}$. Another form of (2.14) is :

$$E\big(\psi(S_{d_t})|\mathcal{F}_t\big) = \psi(S_t)\big(1 - \frac{X_t^+}{S_t}\big) + X_t^+ \int_{S_t}^\infty \frac{\psi(v)}{v^2}\, dv \tag{2.15}$$

Proof of Proposition A.2.

1) Thanks to the scaling property : $S_{g_t} \overset{law}{=} \sqrt{t}\, S_{g_1}$.

On the other hand, for $\alpha > 0$, one has : $P(S_{g_1} < \alpha) = P(g_1 < T_\alpha) = P(1 < d_{T_\alpha})$.
Now,

$$\begin{aligned} d_{T_\alpha} = T_\alpha + T_0 \circ \theta_{T_\alpha} &\overset{(law)}{=} T_\alpha + T'_\alpha \text{ (where } T'_\alpha \text{ is an independent copy of } T_\alpha) \\ &\overset{(law)}{=} T_{2\alpha}. \end{aligned}$$

Therefore :

$$P(S_{g_1} < \alpha) = P(1 < T_{2\alpha}) = P(S_1 < 2\alpha) = P\Big(\frac{1}{2}|N| < \alpha\Big).$$

2) Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, be positive, bounded and with integral equal to 1.
Let $H(x) := \displaystyle\int_0^x h(y)dy$. Let, furthermore, for $s \geq 0$:

$$M_s^h := h(S_{g_s})X_s^+ + h(S_s)(S_s - X_s^+) + 1 - H(S_s). \tag{2.16}$$

The Tanaka-Itô formula and the balayage formula $\big($cf [RY], chap. VI, §4$\big)$ imply that $(M_s^h,\ s \geq\ 0)$ is a positive martingale. We may apply Doob's optimal stopping theorem at the stopping time T_a, since $(M_{s\wedge T_a}^h)$ is bounded ; thus :

$$1 = 1 - H(a) + a\,E\big(h(S_{g_{T_a}})\big).$$

Therefore :

$$E\big[h(S_{g_{T_a}})\big] = \frac{1}{a}\int_0^a h(x)dx, \text{ i.e. (2.10) is proven.}$$

3) We get, successively, for $c > 0$:

$$P(S_{\tau_l} < c) = P(\tau_l < T_c) = P(l < L_{T_c})$$

It is well known that L_{T_c} is distributed as an exponential r.v. with parameter $\dfrac{1}{2c}\cdot$
Indeed, $\Big(h(L_t)X_t^+ - \dfrac{1}{2}\,H(L_t),\ t \geq 0\Big)$ is a martingale ; hence :

$$E\big[h(L_{T_c})c\big] = \frac{1}{2}\,E\big[H(L_{T_c})\big]$$

i.e. L_{T_c} is an exponential r.v. with parameter $\dfrac{1}{2c}\cdot$

4) For $c > 0$ and $a < 0$, we get :

$$P(S_{T_a} < c) = P(T_a < T_c) = \frac{c}{c-a}, \quad \text{hence : } f_{S_{T_a}}(c) = -\frac{a}{(c-a)^2}\,1_{c\geq 0}$$

5) The first assertion of point 5 may be proven similarly to the previous point. Let us show (2.14) and (2.15).
Since $d_t = t + T_0 \circ \theta_t$, we get :

$$\begin{aligned} E\big[\psi(S_{d_t})|\mathcal{F}_t\big] = &\quad E\big[\psi(S_t \vee S_{[t,\,t+T_0\circ\theta_t]})|\mathcal{F}_t\big] \text{ with } S_{[t,\,t+T_0\circ\theta_t]} := \sup_{u\in[t,\,t+T_0\circ\theta_t]} X_u \\ = &\quad \widehat{E}\big[\psi(S_t \vee (X_t + \sup_{0\leq u\leq \widehat{T}_{-X_t}} \widehat{B}_u)\big] \end{aligned}$$

where $(\widehat{B}_u,\ u \geq 0)$ denotes a Brownian motion starting from 0 and independent from $\mathcal{F}_t$. In the preceding expression, the r.v.'s S_t and X_t are frozen and the expectation bears upon $\widehat{B}$.

Therefore :

$$
\begin{aligned}
E\big[\psi(S_{d_t})|\mathcal{F}_t\big] &= \widehat{E}\Big[\psi\big(S_t \vee \frac{X_t^+}{\widehat{U}}\big)\Big] \qquad \big(\text{from } (2.13)\big)\\
&= \int_0^1 \psi\Big(S_t \vee \frac{X_t^+}{u}\Big)du = \int_0^{\frac{X_t^+}{S_t}} \psi\Big(\frac{X_t^+}{u}\Big)du + \int_{\frac{X_t^+}{S_t}}^1 \psi(S_t)du\\
&= X_t^+ \int_{S_t}^\infty \psi(v)\frac{dv}{v^2} + \psi(S_t)\Big(1-\frac{X_t^+}{S_t}\Big)
\end{aligned}
$$

i.e. (2.15) has been proven.

A.1.2. We now prove point 1 of Theorem A.1.

1) We first show that, for every $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ integrable, we have :

$$
E\big[\psi(S_{g_t})\big] \underset{t\to\infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}} \int_0^\infty \psi(x)dx \tag{2.17}
$$

Indeed, from (2.9) :

$$
E\big[\psi(S_{g_t})\big] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{x^2}{2}} \psi\Big(\frac{\sqrt{t}x}{2}\Big)dx = \frac{2\sqrt{2}}{\sqrt{\pi t}} \int_0^\infty e^{-(\frac{x^2}{t})}\psi(x)dx \underset{t\to\infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}} \int_0^\infty \psi(x)dx
$$

2) Let us prove that :

$$
\frac{E\big[\varphi(S_{g_t})|\mathcal{F}_s\big]}{E\big[\varphi(S_{g_t})\big]} \underset{t\to\infty}{\longrightarrow} M_s^\varphi \quad \text{a.s., where } M_s^\varphi \text{ is defined by (2.4).}
$$

We already know, from the preceding point, that $E\big[\varphi(S_{g_t})\big] \underset{t\to\infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}}\cdot$ We then write :

$$
\begin{aligned}
N_t := E\big[\varphi(S_{g_t})|\mathcal{F}_s\big] &= E\big[\varphi(S_{g_t})\, 1_{(g_t<s)}|\mathcal{F}_s\big] + E\big[\varphi\,(S_{g_t})1_{(g_t>s)}|\mathcal{F}_s\big]\\
&:= (1)_t + (2)_t
\end{aligned} \tag{2.18}
$$

and we shall study successively the asymptotic behaviors, as $t \to \infty$ of $(1)_t$ and $(2)_t$.

2) a. Asymptotic behavior of $(1)_t$.

$$
\begin{aligned}
(1)_t = E\big[\varphi(S_{g_t})1_{(g_t<s)}|\mathcal{F}_s\big] &= E\big[\varphi(S_{g_s})1_{(g_t<s)}|\mathcal{F}_s\big]\\
&= \varphi\,(S_{g_s})E[1_{(g_t<s)}|\mathcal{F}_s]
\end{aligned} \tag{2.19}
$$

since $g_t = g_s$ if $g_t < s$. However, since $1_{(g_t<s)} = 1_{(d_s>t)} = 1_{s+T_0\circ\theta_s>t}$, we get :

$$
\begin{aligned}
E\big[1_{(g_t<s)}|\mathcal{F}_s\big] &= E\big[1_{T_0\circ\theta_s>t-s}|\mathcal{F}_s\big] = E_{|X_s|}[T_0 > t-s]\\
&\underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\frac{|X_s|}{\sqrt{t-s}} \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\frac{|X_s|}{\sqrt{t}}
\end{aligned} \tag{2.20}
$$

Hence, gathering (2.19), (2.20) and (2.17) :

$$
\frac{E\big[\varphi(S_{g_t})\, 1_{(g_t<s)}|\mathcal{F}_s\big]}{E\big(\varphi(S_{g_t})\big)} \underset{t\to\infty}{\longrightarrow} \frac{1}{2}\,\varphi(S_{g_s})|X_s| \qquad \text{a.s.} \tag{2.21}
$$

2) b. Asymptotic behavior of $(2)_t$.

$$\begin{aligned}(2)_t &= E\big[\varphi(S_{g_t})\,1_{(g_t>s)}|\mathcal{F}_s\big] = E\big[\varphi(S_s \vee S_{[s,g_t]})\,1_{(g_t>s)}|\mathcal{F}_s\big] \\ \text{with } & S_{[s,g_t]} := \sup_{u\in[s,g_t]} X_u \\ &= \widehat{E}\big[\varphi\big(S_s \vee (X_s + \widehat{S}_{\hat{g}_{t-s}^{(-X_s)}}\big)\,1_{s+\widehat{T}_{-X_s}<t}\big] \qquad (2.22)\end{aligned}$$

with : $g_t^{(a)} := \sup\{u \leq t,\ X_u = a\}$ and where, in (2.22), the expressions without hats are frozen, and where those with hats are being integrated. So, we have to estimate :

$$\widehat{E}\big[\psi(\widehat{S}_{\hat{g}_t^{(a)}}\,1_{\widehat{T}_a<t}\big] \qquad (2.23)$$

where we shall replace t by $t-s$ and, we denote :

$$\psi(c) := \varphi\big(S_s \vee (X_s + c)\big) \qquad (2.24)$$

Thus, we now estimate the asymptotic behavior, as $t \to \infty$, of $E\big(\psi(S_{g_t^{(a)}})\,1_{T_a<t}\big)$ (where we have deleted the hats, which now, are no longer useful), and we distinguish two cases :

$\boxed{\text{Case } 1 :\ a(=-X_s) > 0}$

$$\begin{aligned}E\big[\psi(S_{g_t^{(a)}})\,1_{T_a<t}\big] &= E\big[\psi(a + S_{g_{t-T_a}})\,1_{T_a<t}\big] \\ &\sim \frac{2\sqrt{2}}{\sqrt{\pi t}}\int_0^\infty \psi(a+x)dx \qquad (2.25)\end{aligned}$$

from (2.17). Thus, plugging this estimate (2.25) into (2.22), (and choosing there ψ as given by (2.24)), we obtain :

$$\begin{aligned}\frac{E\big[\varphi(S_{g_t})\,1_{(g_t>s)}1_{X_s<0}|\mathcal{F}_s\big]}{E\big[\varphi(S_{g_t})\big]} &\underset{t\to\infty}{\longrightarrow} 1_{X_s<0}\int_0^\infty \varphi\big(S_s \vee (X_s - X_s + x)\big)dx \\ &= 1_{X_s<0}\int_0^\infty \varphi(S_s\vee x)dx \\ &= 1_{X_s<0}\Big[\int_0^{S_s}\varphi(S_s)dx + \int_{S_s}^\infty \varphi(x)dx\Big] \\ &= 1_{X_s<0}\big[\varphi(S_s)\cdot S_s + 1 - \Phi(S_s)\big] \qquad (2.26)\end{aligned}$$

$\boxed{\text{Case } 2 :\ a(=-X_s) < 0}$

$$E\big[\psi(S_{g_t^{(a)}}\,1_{T_a<t}\big] = E\big[\psi\big(S_{T_a} \vee (a + \widehat{S}_{\hat{g}_{t-T_a}})\big)\,1_{T_a<t}\big] \qquad (2.27)$$

where, in (2.27) $\widehat{S}_{\hat{g}_{t-T_a}}$ is independent from $\mathcal{F}_{T_a}$, and where the expectation is computed relatively to all the variables :

$$\underset{t\to\infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi t}}\,E\Big(\int_0^\infty \psi\big(S_{T_a\vee}(a+x)\big)dx\Big) \qquad (2.28)$$

from (2.17). But, we have :

$$\begin{aligned}
\Delta := \; & E\Big(\int_0^\infty \psi\big(S_{T_a} \vee (a+x)\big)dx = E\Big(\int_a^\infty \psi(S_{T_a} \vee y)dy\Big) \\
= \; & E\big(-a\psi(S_{T_a}) + \int_0^\infty \psi(S_{T_a} \vee y)dy\big) \\
= \; & a^2 \int_0^\infty \psi(c)\frac{dc}{(c-a)^2} + (-a)\int_0^\infty \frac{dc}{(c-a)^2}\Big[c\psi(c) + \int_c^\infty \psi(z)dz\Big]
\end{aligned}$$

$\big($from (2.12)$\big)$

$$= \int_0^\infty \psi(c)\Big[\frac{a^2}{(c-a)^2} - \frac{ac}{(c-a)^2} + \frac{c}{c-a}\Big]dc = \int_0^\infty \psi(c)dc \tag{2.29}$$

Thus, from (2.28) and (2.29) :

$$E\big[\psi\big(S_{g_t^{(a)}}\big)\, 1_{T_a<t}\big] \underset{t\to\infty}{\sim} 2\sqrt{\frac{2}{\pi t}}\int_0^\infty \psi(c)dc \tag{2.30}$$

Bringing this estimate into (2.22), with ψ defined by (2.24), we obtain :

$$\begin{aligned}
&\frac{E\big[\varphi(S_{g_t})\,1_{(g_t>s)}1_{X_s>0}|\mathcal{F}_s\big]}{E\big[\varphi(S_{g_t})\big]} \underset{t\to\infty}{\longrightarrow} 1_{X_s>0}\int_0^\infty \varphi\big(S_s \vee (X_s+c)\big)\,dc \\
&= 1_{X_s>0}\int_{X_s}^\infty \varphi(S_s \vee y)dy = 1_{X_s>0}\big[\varphi(S_s)(S_s - X_s) + \int_{S_s}^\infty \varphi(x)dx\big]
\end{aligned} \tag{2.31}$$

Finally, gathering (2.31), (2.26) and (2.21) leads to :

$$\begin{aligned}
\frac{E\big[\varphi(S_{g_t})|\mathcal{F}_s\big]}{E\big[\varphi(S_{g_t})\big]} \underset{t\to\infty}{\longrightarrow} \; & \frac{1}{2}\varphi(S_{g_s})|X_s| + 1_{X_s<0}\big[\varphi(S_s)S_s + 1 - \Phi(S_s)\big] \\
+ \; & 1_{X_s>0}\big[\varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s)\big] \\
= \; & \frac{1}{2}\varphi(S_{g_s})|X_s| + \varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s) = M_s^\varphi
\end{aligned} \tag{2.32}$$

3) We may now finish the proof of point 1 in Theorem A.1.
The Itô-Tanaka formula and the balayage formula $\big($see [RY], chap. VI, §4$\big)$ imply :

$$M_s^\varphi = 1 + \int_0^s \Big(\frac{1}{2}\varphi(S_{g_u})sgn\, X_u - \varphi(S_u)1_{X_u>0}\Big)dX_u. \tag{2.33}$$

It follows $\big($see[RVY, II] for similar arguments$\big)$, that $(M_s^\varphi,\, s \geq 0)$ is a martingale and that in particular $E(M_s^\varphi) = 1$. Thus, from this latter relation and since $\dfrac{E\big[\varphi(S_{g_t})|\mathcal{F}_s\big]}{E\big[\varphi(S_{g_t})\big]} \underset{t\to\infty}{\longrightarrow} M_s^\varphi$ a.s., this last convergence holds equally in L^1 $\big($cf [M], T. 21$\big)$. Point 1 of Theorem A.1. follows immediately.

A.1.3. We now prove $Q^\varphi[S_\infty = \infty] = \frac{1}{2}\cdot$
We have, for all $\alpha > 0$ and $a > 0$:

$$Q^\varphi(S_a > \alpha) = Q^\varphi(T_\alpha < a) = E[1_{T_\alpha<a}M_a^\varphi] = E[1_{T_\alpha<a}M_{T_\alpha}^\varphi]$$

from Doob's optional stopping theorem. Thus, letting $a \to +\infty$, we obtain :

$$\begin{aligned} Q^\varphi[S_\infty > \alpha] &= E[M^\varphi_{T_\alpha}] = E\Big\{\frac{1}{2}\,\varphi(S_{g_{T_\alpha}})\alpha + 1 - \Phi(\alpha)\Big\} \\ &= \frac{\alpha}{2\alpha}\int_0^\alpha \varphi(x)dx + \int_\alpha^\infty \varphi(x)dx \end{aligned}$$

from point 2 of Proposition A.2. Hence :

$$Q^\varphi[S_\infty > \alpha] \underset{\alpha\to\infty}{\longrightarrow} Q^\varphi[S_\infty = \infty] = \frac{1}{2}\int_0^\infty \varphi(x)dx = \frac{1}{2} \tag{2.34}$$

A.1.4. We now prove that $g := \sup\{t : X_t = 0\}$ is Q^φ a.s. finite.
Let $0 < a < t$. We have :

$$Q^\varphi[g_t > a] = Q^\varphi[d_a < t] = E[1_{d_a<t}\cdot M^\varphi_t] = E[1_{d_a<t}M^\varphi_{d_a}]$$

Hence, since $g_\infty = g$ and letting $t \to +\infty$, we obtain :

$$\begin{aligned} Q^\varphi[g > a] &= \lim_{t\to\infty} E[1_{d_a<t}M^\varphi_{d_a}] = E[M^\varphi_{d_a}] \\ &= E\big[\varphi(S_{d_a})S_{d_a} + 1 - \Phi(S_{d_a})\big] \end{aligned} \tag{2.35}$$

We shall show now that $Q^\varphi[g > a] \underset{a\to\infty}{\longrightarrow} 0$, which proves that g is Q^φ a.s. finite.
But : $E\big[1 - \Phi(S_{d_a})\big] \underset{a\to\infty}{\longrightarrow} 0$ from the dominated convergence Theorem.
On the other hand, from (2.15) :

$$\begin{aligned} E\big[\varphi(S_{d_a})S_{d_a}\big] &= E\Big[\varphi(S_a)(S_a - X_a^+) + X_a^+\int_{S_a}^\infty \frac{\varphi(v)v}{v^2}\,dv\Big] \\ &\le E\big[\varphi(S_a)S_a\big] + E\Big\{\frac{X_a^+}{S_a}\int_{S_a}^\infty \varphi(v)dv\Big\} \\ &\le E\Big\{\big[\varphi(S_a)S_a\big] + 1 - \Phi(S_a)\Big\} \end{aligned}$$

But : $E\big(1 - \Phi(S_a)\big) \underset{a\to\infty}{\longrightarrow} 0$ from the dominated convergence Theorem and :

$$\begin{aligned} E\big[\varphi(S_a)\cdot S_a\big] &= \sqrt{\frac{2}{\pi a}}\int_0^\infty \varphi(x)x\,e^{-\frac{x^2}{2a}}dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(x)\Big[\frac{x}{\sqrt{a}}\,e^{-\frac{1}{2}\left(\frac{x}{\sqrt{a}}\right)^2}\Big]dx \underset{a\to\infty}{\longrightarrow} 0 \end{aligned}$$

because $\dfrac{x}{\sqrt{a}}\,e^{-\frac{1}{2}\left(\frac{x}{\sqrt{a}}\right)^2}$ is uniformly bounded and converges to 0 as $a \to \infty$.
Thus, from (2.35) :

$$Q^\varphi[g = \infty] = \lim_{a\to\infty} Q^\varphi(g > a) = \lim_{a\to\infty} E\big[\varphi(S_{d_a})S_{d_a} + 1 - \Phi(S_{d_a})\big] = 0 \tag{2.36}$$

A.1.5. Computation of Azéma's supermartingale $Z_t := Q^\varphi(g > t|\mathcal{F})$.
In order to complete the proof of Theorem A.1., we shall use the technique of enlargement of filtration, i.e. : we shall work within the filtration $(\mathcal{G}_t,\, t \ge 0)$, where $(\mathcal{G}_t,\ t \ge 0)$ is the

smallest filtration with contains $(\mathcal{F}_t,\, t \geq 0)$ and such that $g := \sup\{t\ ;\ X_t = 0\}$ becomes a $(\mathcal{G}_t,\, t \geq 0)$ stopping time. To apply the enlargement formula, we need to compute the Azéma supermartingale : $Z_t := Q^\varphi(g > t|\mathcal{F}_t)$

Lemma A.3.

$$1)\ Z_t := Q^\varphi(g > t|\mathcal{F}_t) = \frac{\varphi(S_t)(S_t - X_t^+) + 1 - \Phi(S_t)}{M_t^\varphi} \tag{2.37}$$

2)*For every positive, $(\mathcal{F}_s)$ predictable process, $(K_s,\, s \geq 0)$ one has :*

$$E_{Q^\varphi}[K_g] = \frac{1}{2}\, E\Big(\int_0^\infty K_s \varphi(S_s) dL_s\Big) \tag{2.38}$$

Proof of Lemma A.3.

1) Since $g := \sup\{t \geq 0\ ;\ X_t = 0\}$, we get :

$$Q^\varphi[g > t|\mathcal{F}_t] = E_{Q^\varphi}[1_{d_t < \infty}|\mathcal{F}_t] = \frac{1}{M_t^\varphi}\, E[M_{d_t}^\varphi|\mathcal{F}_t]$$

$$= \frac{1}{M_t^\varphi}\, E\big[\big(\varphi(S_{d_t})S_{d_t} + 1 - \Phi(S_{d_t})\big)|\mathcal{F}_t\big] \tag{2.39}$$

Applying (2.15) to the function $\psi(x) := \varphi(x)x + 1 - \Phi(x)$, an elementary computation leads to :

$$E\big[\varphi(S_{d_t})S_{d_t} + 1 - \Phi(S_{d_t})|\mathcal{F}_t\big] = \varphi(S_t)(S_t - X_t^+) + 1 - \Phi(S_t)$$

hence (2.37), by using (2.39).

2) From (2.37), we deduce that for every bounded $(\mathcal{F}_t,\, t \geq 0)$ stopping time T :

$$\begin{aligned} E_{Q^\varphi}[1_{[0,T]}(g)] &= E_{Q^\varphi}[1 - 1_{g>T}] = E\big[M_T^\varphi - \varphi(S_T)(S_T - X_T^+) + (1 - \Phi(S_T)\big] \\ &= \frac{1}{2} E\big[\varphi(S_{g_T})|X_T|\big] \qquad \text{from (2.4)} \\ &= \frac{1}{2}\, E\Big(\int_0^\infty 1_{(s<T)} \varphi(S_s) dL_s\Big) \end{aligned} \tag{2.40}$$

from the balayage formula. Then, we extend the result from the elementary processes $1_{[0,T]}(s)$ to every positive $(\mathcal{F}_s)$, predictable process (K_s) by using the monotone class theorem. Thus :

$$E_{Q^\varphi}[K_g] = \frac{1}{2}\, E\Big(\int_0^\infty K_s \varphi(S_s) dL_s\Big) \qquad \square$$

A.1.6. We now prove points 2ii, 2iii, and 3 iii of Theorem A.1.

1. Applying (2.38) with $K_s = f_1(L_s) f_2(S_s)$, with f_1, f_2 Borel and positive, we obtain :

$$\begin{aligned} E_{Q^\varphi}\big[f_1(L_g) f_2(S_g)\big] &= \frac{1}{2}\, E\Big(\int_0^\infty f_1(L_s) f_2(S_s) \varphi(S_s) dL_s\Big) \\ &= \frac{1}{2}\, E\Big(\int_0^\infty f_1(v) f_2(S_{\tau_v}) \varphi(S_{\tau_v}) dv\Big) \end{aligned}$$

(after the change of variables $L_s = v$).

$$= \frac{1}{2} \int_0^\infty \int_0^\infty f_1(v) f_2(c) \varphi(c) \frac{v}{2c^2}\, e^{-\frac{v}{2c}} dc\, dv \tag{2.41}$$

with the help of point 3 of Proposition A.2. Thus, the density of the r.v. (L_g, S_g) under Q^φ equals :

$$f^{Q^\varphi}_{L_g,S_g}(v,c) = \frac{1}{4}\,\frac{v}{c^2}\,e^{-\frac{v}{2c}}\varphi(c)1_{v\geq 0}1_{c\geq 0} \tag{2.42}$$

Points 2ii, 2iii of Theorem A.1. follow easily from this formula $\big($with the help of (2.34)$\big)$.

2. To show point 3iii of Theorem A.1., we use (2.38) with:
$K_s := F(X_u,\, u\leq s)\, f_1(L_s)f_2(S_s)$. We obtain :

$$\begin{aligned}
E_{Q^\varphi}\big[F(X_u,\, u\leq g)f_1(L_g)f_2(S_g)\big] &= \frac{1}{2}\,E\Big(\int_0^\infty F(X_u,\, u\leq s)f_1(L_s)f_2(S_s)d\,L_s\Big)\\
&= \frac{1}{2}\,E\Big(\int_0^\infty F(X_u\, u\leq \tau_v)f_1(v)f_2(S_{\tau_v})\varphi(S_{\tau_v})dv\Big)
\end{aligned}$$

(after making the change of variables : $L_s = v$)

$$= \frac{1}{2}\int_0^\infty E\big(F\big((X_u,\, u\leq\tau_v)\big)\big|S_{\tau_v}=c\big)f_1(v)f_2(c)\varphi(c)\frac{v}{2c^2}\,e^{-\frac{v}{2c}}dc\,dv \tag{2.43}$$

But, it also holds that :

$$\begin{aligned}
&E_{Q^\varphi}\big[F(X_u,\, u\leq g)f_1(L_g)f_2(S_g)\big]\\
&= \int_0^\infty\int_0^\infty E_{Q^\varphi}\big[F(X_u,\, u\leq g)|L_g=v, S_g=c\big]f_1(v)f_2(c)\frac{v}{4c^2}\,e^{-\frac{v}{2c}}\varphi(c)dc\,dv
\end{aligned} \tag{2.44}$$

Hence, comparing (2.43) and (2.44), we obtain :

$$E_{Q^\varphi}\big[F(X_u,\, u\leq g)|L_g=v,\, S_g=c\big] = E\big(F(X_u,\, u\leq\tau_v)|S_{\tau_v}=c\big) \tag{2.45}$$

which is point 3 *iii* of Theorem A.1.

A.1.7. End of the proof of Theorem A.1. with the help of enlargement formulae. From Girsanov's theorem $\big($cf [RY], chap. VIII, §3$\big)$, using the expression (2.5) of M_t^φ as a stochastic integral, we know that there exists a $\big((\mathcal{F}_t)_{t\geq 0},\, Q^\varphi\big)$ Brownian motion $(\beta_t,\, t\geq 0)$ such that :

$$X_t = \beta_t + \int_0^t \frac{\frac{1}{2}\varphi(S_{g_s})sgn\,(X_s) - \varphi(S_s)1_{X_s>0}}{M_s^\varphi}\,ds \tag{2.46}$$

We denote by $(\mathcal{G}_t,\, t\geq 0)$ the smallest filtration which contains $(\mathcal{F}_t,\, t\geq 0)$ and which makes g a $(\mathcal{G}_t,\, t\geq 0)$ stopping time. The enlargement formulae $\big($see [J], [JY], or [MY]$\big)$ imply the existence of a $\big((\mathcal{G}_t,\, t\geq 0),\, Q^\varphi\big)$ brownian motion $(\widetilde{\beta}_t,\, t\geq 0)$ such that :

$$\begin{aligned}
X_t &= \widetilde{\beta}_t + \int_0^t \frac{\frac{1}{2}\varphi(S_{g_s})sgn\,(X_s) - \varphi(S_s)1_{X_s>0}}{M_s^\varphi}\,ds\\
&\quad + \int_0^{t\wedge g}\frac{d<Z,\,X>_s}{Z_s} - \int_{t\wedge g}^t \frac{d<Z,\,X>_s}{1-Z_s}
\end{aligned} \tag{2.47}$$

In order to make (2.47) more explicit, we need to compute the martingale part $\widetilde{Z}_t$ in the Doob-Meyer decomposition of Z_t relatively to $\big((\mathcal{F}_t),\, Q\big)$. From Itô's formula and (2.37), we get :

$$d\widetilde{Z}_t = -\frac{\varphi(S_t)(S_t - X_t^+) + \big(1 - \Phi(S_t)\big)}{M_t^2}\Big[\frac{1}{2}\varphi(S_{g_t}) sgn\, X_t\, dX_t - \varphi(S_t) 1_{X_t>0} dX_t\Big]$$
$$-\frac{1}{M_t}\,\varphi(S_t)1_{X_t>0}\, dX_t + (\text{b.v. term}) \tag{2.48}$$

(b.v. = bounded variation).
Thus, the bracket $< Z, X > = < \widetilde{Z}, X >$ satisfies :

$$d < Z, X >_t = -\frac{\varphi(S_t)(S_t - X_t^+) + \big(1 - \Phi(S_t)\big)}{M_t^2}\Big[\frac{1}{2}\,\varphi(S_{g_t})\, sgn\, X_t - \varphi(S_t)1_{X_t>0}\Big] dt$$
$$-\frac{1}{M_t}\,\varphi(S_t)1_{X_t>0}\, dt. \tag{2.49}$$

as it may be computed indifferently under P or Q.
Thus, plugging (2.49) in (2.47) $\big($and using (2.37)$\big)$, we obtain, for all $t \geq 0$:

$$X_{g+t} = \big(\widetilde{\beta}_{g+t} - \widetilde{\beta}_g\big) + \int_g^{g+t} \frac{\frac{1}{2}\varphi(S_{g_s})\, sgn\, X_s - \varphi(S_s)1_{X_s>0}}{M_s}\, ds$$
$$+ \int_g^{g+t} \Big[\frac{\varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s)}{M_s^2}\Big\{\frac{1}{2}\varphi(S_{g_s}) sgn\, X_s - \varphi(S_s)1_{X_s>0}\Big\}$$
$$+\frac{1}{M_s}\varphi(S_s)1_{X_s>0}\Big] \cdot \frac{2M_s}{\varphi(S_{g_s})|X_s|}\, ds. \tag{2.50}$$

We obtain, after simplification :

$$\begin{aligned} X_{g+t} = \big(\widetilde{\beta}_{g+t} - \widetilde{\beta}_g\big) &+ \int_g^{g+t} \frac{1}{M_s\varphi(S_{g_s})|X_s|}\big\{\varphi(S_{g_s}) sgn\, X_s M_s - 2\varphi(S_s)1_{X_s>0}M_s\big\} ds \\ &+ \int_g^{g+t} \frac{2\varphi(S_s)1_{X_s>0}}{\varphi(S_{g_s})|X_s|}\, ds, \qquad \text{that is :} \\ X_{g+t} &= \big(\widehat{\beta}_{g+t} - \widehat{\beta}_g\big) + \int_0^t \frac{ds}{X_{g+s}}\cdot \end{aligned} \tag{2.51}$$

On the other hand, the sign of X_u is constant after g : it is positive with probability 1/2, from point 3iii of Theorem A.1.
Thus, the equation (2.51) shows that, with probability 1/2, $(X_{g+t},\ t \geq 0)$ is a Bessel process with dimension 3 (and, with probability 1/2, it is the opposite of a 3-dimensional Bessel process).
The independence of $(X_u,\ u \leq g)$ and of $(X_u,\ u \geq g)$ follows from the fact that the equation (2.51) admits a unique strong solution.
Finally, we note that (2.47), written before g, leads to :

$$X_t = \widetilde{\beta}_t - \int_0^t \frac{\varphi(S_s)1_{X_s>0}}{\varphi(S_s)(S_s - X_s^+) + 1 - \Phi(S_s)}\, ds \tag{2.52}$$

A.1.8. We now prove point 4 of Theorem A.1.
Since, owing to point 3iii, conditionally to $L_g = v$ and $S_g = c$, $(X_t, t \leq g)$ is a (stopped) Brownian motion, from Pitman's theorem $\big($see [P]$\big)$, the process $\big(|X_t| + L_t, t \geq 0\big)$ is a 3-dimensional Bessel process.
From 3,iii, the same process, after g, is also a 3-dimensional Bessel process ; note that, then $d\big(|X_t| + L_t\big) = d\big(|X_t|\big)$. Thus, the entire process $\big(|X_t| + L_t, t \geq 0\big)$ is a 3-dimensional Bessel process independent from (S_g, L_g) since the conditional law of $\big(|X_t| + L_t, t \geq 0\big)$ does not depend on (S_g, L_g).

A.1.9. Remark A.4.
In [RVY, II], we penalized Brownian motion with $\varphi(S_t)$, i.e. we "favored" the Brownian trajectories which are not "too high", and it followed that $Q^\varphi(S_\infty < \infty) = 1$; in fact, under Q^φ, the trajectories decided to go to $-\infty$ as $t \longrightarrow \infty$". It is their "response" to that kind of penalisation. What is happening here ?
We have penalised by $\varphi(S_{g_t})$, i.e. : favored the trajectories which are not too high before their last zero. How will the trajectories "respond" ? Will they decide to remain bounded ? Or to have a last zero ? We shall show that the trajectories "decide", under Q^φ, to eventually quit 0, forever, so that $g < \infty \quad Q^\varphi$ a.s. hence $S_g < \infty$ a.s., whereas $S_\infty = \infty$ with probability 1/2.

3 Part A. 2. Penalisation by $\varphi(S_{d_t})$.

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ denote a probability density, i.e. :

$$\int_0^\infty \varphi(x)dx = 1. \tag{3.1}$$

As previously, we denote : $\Phi(x) := \displaystyle\int_0^x \varphi(y)dy \quad (x \geq 0)$.
We define $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$ via :

$$f(b, a) := \varphi(b)\Big(1 - \frac{a_+}{b}\Big) + a_+ \int_b^\infty \frac{\varphi(v)}{v^2}\,dv \tag{3.2}$$

Theorem A.4. *Under the previous hypothesis (3.1), one has for any $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$:*

$$\begin{aligned}
\lim_{t\to\infty} \frac{E\big[1_{\Lambda_s}\varphi(S_{d_t})\big]}{E\big[\varphi(S_{d_t})\big]} &= \lim_{t\to\infty} \frac{E\big[1_{\Lambda_s} f(S_t, X_t)\big]}{E\big[f(S_t, X_t)\big]} = \lim_{t\to\infty} \frac{E\big[1_{\Lambda_s}\varphi(S_t)\big]}{E\big[\varphi(S_t)\big]} && (3.3)\\
&= E[1_{\Lambda_s}\widetilde{M}_s^\varphi] := \widetilde{Q}^\varphi(\Lambda_s) && (3.4)\\
\textit{with } \widetilde{M}_s^\varphi : &= \varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s) && (3.5)
\end{aligned}$$

where $(\widetilde{M}_s^\varphi, s \geq 0)$ is a $\big(P, (\mathcal{F}_s, s \geq 0)\big)$ positive martingale.
In other terms, the penalisation by $\varphi(S_{d_t})$ is the same as that by $\varphi(S_t)$ $\big($see (1.1) and (1.2) above, or [RVY, II]$\big)$. Thus, we may refer the reader to [RVY, II] for a study of the canonical process $(X_t, t \geq 0)$ under $\widetilde{Q}^\varphi$.

A.2.1. Proof of Theorem A.4.

1) Recall that, from (2.15) :

$$E\big[\varphi(S_{d_t})|\mathcal{F}_t\big] = \varphi(S_t)\Big(1 - \frac{X_t^+}{S_t}\Big) + X_t^+ \int_{S_t}^{\infty} \frac{\varphi(v)}{v^2}\,dv = f(S_t, X_t) \tag{3.6}$$

which proves the first equality in (3.3).

2) We now study the denominator in (3.3) and we prove that :

$$E\big[f(S_t, X_t)\big] \underset{t\to\infty}{\sim} E\big[\varphi(S_t)\big] \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_0^{\infty} \varphi(x)dx = \sqrt{\frac{2}{\pi t}}, \tag{3.7}$$

To prove (3.7), we study successively the 3 terms which constitute $E\big[f(S_t, X_t)\big]$:

- $$E\big[\varphi(S_t)\big] = \sqrt{\frac{2}{\pi t}} \int_0^{\infty} \varphi(x)\, e^{-\frac{x^2}{2t}}\, dx \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_0^{\infty} \varphi(x)\, dx = \sqrt{\frac{2}{\pi t}}\cdot \tag{3.8}$$

We now prove that :

- $$E\left[\varphi(S_t)\frac{X_t^+}{S_t}\right] = o\Big(\frac{1}{\sqrt{t}}\Big) \qquad (t \to \infty) \qquad \text{and} \tag{3.9}$$
- $$E\left[X_t^+ \int_{S_t}^{\infty} \frac{\varphi(v)}{v^2}\, dv\right] = o\Big(\frac{1}{\sqrt{t}}\Big) \qquad (t \to \infty) \tag{3.10}$$

(3.9) and (3.10) are obvious consequences of Lemma A.5, because :

$$\begin{aligned}
&\varphi(S_t)\frac{X_t^+}{S_t} \le \varphi(S_t) 1_{X_t \ge 0} \qquad \text{and} \\
&X_t^+ \int_{S_t}^{\infty} \frac{\varphi(v)}{v^2}\, dv \le \frac{X_t^+}{S_t} \int_{S_t}^{\infty} \frac{\varphi(v)}{v}\, dv \le 1_{X_t \ge 0}\, \widetilde{\varphi}(S_t), \quad \text{with} \\
&\widetilde{\varphi}(c) = \int_c^{\infty} \frac{\varphi(v)}{v}\, dv \quad ;\ \widetilde{\varphi} \text{ is integrable since :} \\
&\int_0^{\infty} \widetilde{\varphi}(c)dc = \int_0^{\infty} dc \int_c^{\infty} \frac{\varphi(v)}{v}\, dv = \int_0^{\infty} \frac{\varphi(v)}{v} \int_0^{v} dc = \int_0^{\infty} \varphi(v)dv.
\end{aligned}$$

Lemma A.5. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be integrable. Then, for any $\sigma \ge 0$ and $x \le \sigma$:*

$$E_0\big[h(\sigma \vee (x + S_t))\, 1_{x + X_t > 0}\big] = o\Big(\frac{1}{\sqrt{t}}\Big) \tag{3.11}$$

Proof of Lemma A.5.

• For $\sigma = x = 0$ we have, from the formula $\big($see [KS], p. 95$\big)$ which gives the law of the pair (S_t, X_t) :

$$\begin{aligned}
E_0\big[h(S_t) 1_{X_t > 0}\big] &= \sqrt{\frac{2}{\pi t^3}} \int_0^{\infty} db\, h(b) \int_0^{b} (2b - a)\, e^{-\frac{(2b-a)^2}{2t}} da \\
&= \sqrt{\frac{2}{\pi t}} \int_0^{\infty} h(b)db[e^{-\frac{b^2}{2t}} - e^{-\frac{2b^2}{t}}] = o\Big(\frac{1}{\sqrt{t}}\Big)
\end{aligned}$$

from the dominated convergence Theorem.

• For $x \le \sigma,\ \sigma \ge 0$:

$$E_0\big[h(\sigma \vee (x+S_t)) \cdot 1_{x+X_t>0}\big] = h(\sigma)P_0[S_t < \sigma - x,\ X_t > -x] + E\big[h(x+S_t)1_{S_t>\sigma-x}1_{X_t>-x}\big]$$

$$= h(\sigma)\sqrt{\frac{2}{\pi t^3}}\int_0^{\sigma-x} db \int_{-x\wedge b}^{b}(2b-a)e^{-\frac{(2b-a)^2}{2t}}da$$

$$+\sqrt{\frac{2}{\pi t^3}}\int_{\sigma-x}^{\infty} db\, h(x+b)\int_{-x\wedge b}^{b}(2b-a)e^{-\frac{(2b-a)^2}{2t}}da$$

$$= h(\sigma)\sqrt{\frac{2}{\pi t}}\int_0^{\sigma-x} db\big[e^{-\frac{b^2}{2t}} - e^{-\frac{(2b-(-x\vee b))^2}{2t}}\big]$$

$$+\sqrt{\frac{2}{\pi t}}\int_{\sigma-x}^{\infty} h(x+b)db\,\big[e^{-\frac{b^2}{2t}} - e^{-\frac{(2b-(-x\wedge b))^2}{2t}}\big] = o\Big(\frac{1}{\sqrt{t}}\Big)$$

by the dominated convergence Theorem. ■

3) We prove that, for fixed s :

$$E\big[\varphi(S_{d_t})\big|\mathcal{F}_s\big] = E\big[f(S_t,X_t)\big|\mathcal{F}_s\big] \underset{t\to\infty}{\sim} E\big[\varphi(S_t)\big|\mathcal{F}_s\big]. \tag{3.12}$$

The first equality in (3.12) follows immediately from (3.6). Furthermore, from (3.2) we deduce that :

$$\begin{aligned} E\big[f(S_t,X_t)\big|\mathcal{F}_s\big] &= E\big[\varphi(S_t)\big|\mathcal{F}_s\big] - E\Big[\varphi(S_t)\frac{X_t^+}{S_t}\big|\mathcal{F}_s\Big] + E\Big[X_t^+\int_{S_t}^{\infty}\frac{\varphi(v)}{v^2}\,dv\big|\mathcal{F}_s\Big] \\ &\equiv (1)_t - (2)_t + (3)_t \end{aligned}$$

we know (see [RVY, II]) that :

$$(1)_t = E\big[\varphi(S_t)\big|\mathcal{F}_s\big] \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}}\big[\varphi(S_s)(S_s - X_s) + \big(1-\Phi(S_s)\big)\big] \tag{3.13}$$

and

$$(2)_t = E\Big[\varphi(S_t)\frac{X_t^+}{S_t}\big|\mathcal{F}_s\Big] = E\Big[\varphi\big(\sigma\vee(x+S_{t-s})\big)\frac{(x+X_{t-s})^+}{\sigma\vee(x+S_{t-s})}\Big] \tag{3.14}$$

with $\sigma = S_s$ and $x = X_s$, and :

$$(2)_t = o\Big(\frac{1}{\sqrt{t-s}}\Big) \qquad \text{from Lemma A.5.} \tag{3.15}$$

The same argument leads to :

$$(3)_t = o\Big(\frac{1}{\sqrt{t-s}}\Big)\cdot \tag{3.16}$$

Finally, gathering (3.14), (3.15) and (3.16) we have obtained :

$$\frac{E\big[\varphi(S_{d_t})\big|\mathcal{F}_s\big]}{E\big[\varphi(S_{d_t})\big]} \underset{t\to\infty}{\longrightarrow} \widetilde{M}_s^{\varphi} = \varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s) \tag{3.17}$$

Itô's formula allows to see that $(\widetilde{M_s^\varphi},\, s \geq 0)$ is a martingale such that $E(\widetilde{M_s^\varphi}) = 1$, which implies, $\big($cf [M], p. 37, t. 21$\big)$ that the convergence in (3.17) takes place in L^1, and Theorem A.4. follows immediately.
Remark By comparison, on one hand, of :
• Theorem A.4., Theorem A.1. and Theorem 4.6. in [RVY, II] ;
• Theorem III. 1., Theorem IV.1. and Theorem IV.2. of [RVY, VII], we obtain the informal, but remarkable following analogy :
• Penalisations by $\varphi(S_{d_t})$ and $\varphi(S_t)$ are identical and differ from the penalisation by $\varphi(S_{g_t})$;
• Penalisations by $\varphi(V_{d_t}^{(1)})$ and $\varphi(V_t^{(1)})$ are identical and differ from the penalisation by $\varphi(V_{g_t}^{(1)})$.

4 Part A. 3. Penalisation by $\varphi(X^*_{g_t})$

A.3.1. We note :

$$X_t^* := \sup_{s \leq t} |X_s| \tag{4.1}$$

$$\text{and, for } a \geq 0 \qquad T_a^* := \inf\{t \geq 0\ ;\ |X_t| = a\} \tag{4.2}$$

As above, we assume that φ is a probability density on $\mathbb{R}_+$; we define :

$$\Phi(x) := \int_0^x \varphi(y)\, dy, \qquad \text{so that } : \ \Phi(0) = 0, \qquad \text{and } \Phi(\infty) = 1.$$

Theorem A.6. *Under the preceding hypotheses, one has :*
1) *For any $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t \to \infty} \frac{E\big[1_{\Lambda_s} \varphi(X^*_{g_t})\big]}{E\big[\varphi(X^*_{g_t})\big]} = E(1_{\Lambda_s} M_s^{*\varphi}) := Q^{*,\varphi}(\Lambda_s) \tag{4.3}$$

with

$$M_s^{*\varphi} := \varphi(X^*_{g_s})|X_s| + \varphi(X_s^*)\big(X_s^* - |X_s|\big) + 1 - \Phi(X_s^*) \tag{4.4}$$

Furthermore, $(M_s^{*\varphi},\, s \geq 0)$ is a positive martingale, which goes to 0 as $s \to \infty$.
2) *Formula (4.4) induces a probability on $Q^{*\varphi}$ on $(\Omega, \mathcal{F}_\infty)$. Under $Q^{*\varphi}$, the canonical process satisfies :*

$$i)\ g := \sup\{t,\, X_t = 0\} \quad \text{is finite a.s.} \tag{4.5}$$

$$ii)\ X_\infty^* = \infty \qquad a.s. \tag{4.6}$$

iii) The processes $(X_t,\, t < g)$ and $(X_{g+t},\, t \geq 0)$ are independent ;

vi) $(X_{g+t},\, t \geq 0)$ is with probability $1/2$, a 3-dimensional Bessel process, starting from 0, and with probability $1/2$, it is the opposite of a 3-dimensional Bessel process.

v) Conditionally on $L_g = v$ and $|X_g^*| = c$, the process $(X_t,\, t \leq g)$ is a Brownian motion stopped at τ_v and conditioned on $X^*_{\tau_v} = c$.

A.3.2 A Lemma for the proof of Theorem A.6.
This proof is close to that of Theorem A.1. Hence, we shall not develop it entirely, and we shall only indicate briefly the elements which differ :
Lemma A.7.
1) For any real a and $\alpha > 0$:

$$\begin{aligned} P_a(X^*_{g_t} < \alpha) &= 0 \qquad \text{if } \alpha < |a| & (4.7) \\ &\underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\,\frac{\alpha}{\sqrt{t}} \qquad \text{if } \alpha > |a| & (4.8) \end{aligned}$$

2) For every Borel and integrable function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ we have :

$$E_a\big(\psi(X^*_{g_t})\big) \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_{|a|}^{\infty} \psi(x)\,dx \tag{4.9}$$

Sketch of proof of Lemma A.7.
(4.7) is obvious. Let us prove (4.8). From the identities :

$$(X^*_{g_t} < \alpha) = (g_t < T^*_\alpha) = (t < d_{T^*_\alpha}) = (t < T^*_\alpha + T_0 \circ \theta_{T^*_\alpha})$$

we deduce :

$$\begin{aligned} \int_0^\infty e^{-\lambda t} P_a(X^*_{g_t} < \alpha)dt &= E_a\Big(\int_0^{d_{T^*_\alpha}} e^{-\lambda t} dt\Big) = \frac{1}{\lambda}\big(1 - E_a(e^{-\lambda(T^*_\alpha + T_0\circ\theta_{T^*_\alpha})})\big) \\ &= \frac{1}{\lambda}\Big\{1 - e^{-\alpha\sqrt{2\lambda}}\,\frac{\cosh\big(a\sqrt{2\lambda}\big)}{\cosh\big(\alpha\sqrt{2\lambda}\big)}\Big\} \qquad \big(\text{see [K, S], p.100}\big) \\ &\underset{\alpha\to 0}{\sim} \frac{\sqrt{2}}{\sqrt{\lambda}}\,\alpha \end{aligned}$$

Hence (4.8) follows, with the help of the Tauberian Theorem (see [Fel], vol. 2, p. 442). Relation (4.9) follows easily from (4.8).
A.3.3. We now prove that :

$$\frac{E\big[\varphi(X^*_{g_t})|\mathcal{F}_s\big]}{E\big(\varphi(X^*_{g_t})\big)} \underset{t\to\infty}{\longrightarrow} M^{*\varphi}_s \qquad \text{a.s.} \tag{4.10}$$

We already note that, from (4.9),

$$E\big(\varphi(X^*_{g_t})\big) \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}} \tag{4.11}$$

On the other hand :

$$\begin{aligned} E\big[\varphi(X^*_{g_t})|\mathcal{F}_s\big] &= E\big[\varphi(X^*_{g_t})1_{g_t<s}|\mathcal{F}_s\big] + E\big[\varphi(X^*_{g_t})1_{g_t>s}|\mathcal{F}_s\big] \\ &= (1)_t + (2)_t & (4.12) \end{aligned}$$

One has :

$$(1)_t = E\big[\varphi(X^*_{g_t})\,1_{g_t<s}|\mathcal{F}_s\big] = E\big[\varphi(X^*_{g_s})\,1_{g_t<s}|\mathcal{F}_s\big]$$

since $g_s = g_t$ when $g_t < s$

$$= \varphi(X^*_{g_s}) E\big[1_{g_t<s}|\mathcal{F}_s\big] \underset{t\to\infty}{\sim} \varphi(X^*_{g_s})|X_s|\sqrt{\frac{2}{\pi(t-s)}} \tag{4.13}$$

from (2.20). On the other hand :

$$\begin{aligned}
(2)_t &= E\big[\varphi(X^*_{g_t})1_{g_t>s}|\mathcal{F}_s\big] = E\big[\varphi(X^*_s \vee X^*_{[s,g_t]})1_{g_t>s}|\mathcal{F}_s\big] \\
&\qquad \Big(\text{with } X^*_{[s,g_t]} = \sup_{u\in[s,g_t]} |X_u|\Big) \\
&\underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}} \int_{|X_s|}^{\infty} \varphi(X^*_s \vee x)\, dx \qquad \text{from (4.9)} \\
&= \sqrt{\frac{2}{\pi t}} \Big\{ \int_{|X_s|}^{X^*_s} \varphi(X^*_s) dx + \int_{X^*_s}^{\infty} \varphi(x) dx \Big\} \\
&= \sqrt{\frac{2}{\pi t}} \Big(\big(X^*_s - |X_s|\big)\varphi(X^*_s) + 1 - \Phi(X^*_s) \Big) \qquad (4.14)
\end{aligned}$$

Gathering (4.11), (4.13) and (4.14), (4.10) follows immediately. Using similar arguments as in proof of Theorem A.4. point 1 of Theorem A.6 follows.

A.3.4. We prove that $Q^{*\varphi}(g < \infty) = 1$.

We have :

$$\begin{aligned}
Q^{*\varphi}(g_t > a) &= Q^{*\varphi}\{d_a < t\} = E[1_{d_a<t} \cdot M^{*\varphi}_t] \\
&= E[1_{d_a<t} M^{*\varphi}_{d_a}] = E\Big[1_{d_a<t}\big[\varphi(X^*_{d_a})X^*_{d_a} + 1 - \Phi(X^*_{d_a})\Big]
\end{aligned}$$

Hence, letting $t \to +\infty$:

$$Q^{*\varphi}(g > a) = E\big[\varphi(X^*_{d_a})X^*_{d_a} + 1 - \Phi(X^*_{d_a})\big] \le 2\, E\big[\varphi(S_{d_a})S_{d_a} + 1 - \Phi(S_{d_a})\big]$$

because, with obvious notations, $X^*_{d_a} = S_{d_a}$ or $-I_{d_a}$ and then

$$Q^{*\varphi}(g = \infty) = \lim_{a\to\infty} Q^{*\varphi}(g > a) = 0 \qquad \text{from (2.36)}$$

A.3.5. We prove that $Q^{*\varphi}(X^*_\infty = \infty) = 1$.

Indeed, operating as above, with $a > 0$, we obtain :

$$\begin{aligned}
Q^{*\varphi}[X^*_\infty > a] &= Q^{*\varphi}[T^*_a < \infty] = E[M^{*\varphi}_{T^*_a}] \\
&= E\big[\varphi(X^*_{g_{T^*_a}})a + 1 - \Phi(a)\big] \\
&= \int_a^\infty \varphi(x) dx + \frac{a}{a} \int_0^a \varphi(x) dx = 1 \qquad (4.15)
\end{aligned}$$

since the r.v., $X^*_{g_{T^*_a}}$ is uniformly distributed on $[0, a]$, as may be seen by using the balayage formula. $\big($see [RY], Chap. VI$\big)$.

It now remains to let a tend to $+\infty$ in (4.15).

We leave to the interested reader the task of completing the proof of Theorem A.6. □

We have, in Section A.1, studied the penalisation by $\varphi(S_{g_t})$. Our technics allows to study penalisations by $\varphi(S_{g_t})1_{X_t>0} + \psi(S_{g_t})1_{X_t<0}$. More precisely :

Proposition A.8. *Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel and integrable. Then, for any $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:*

$$\frac{E_0\big[1_{\Lambda_s}\big(\varphi(S_{g_t})1_{X_t>0} + \varphi(S_{g_t})1_{X_t<0}\big]}{E_0\big[\varphi(S_{g_t})1_{X_t>0} + \varphi(S_{g_t})1_{X_t<0}\big]} \underset{t\to\infty}{\longrightarrow} E_0[1_{\Lambda_s} M_s^{\varphi,\psi}] \tag{4.16}$$

with :

$$M_s^{\varphi,\psi} := \frac{1}{\int_0^\infty (\varphi+\psi)(y)dy}\ \big[\varphi(S_{g_s})X_s + \widehat{M}_s^{\varphi+\psi}\big] \tag{4.17}$$

where :

$$\widehat{M}_s^{\varphi+\psi} := (\varphi+\psi)(S_s)(S_s - X_s) + \int_{S_s}^\infty (\varphi+\psi)(y)dy. \tag{4.18}$$

In particular, for $\psi \equiv 0$ and $\displaystyle\int_0^\infty \varphi(y)dy = 1$

$$\frac{E_0\big[1_{\Lambda_s}\varphi(S_{g_t})1_{X_t>0}\big]}{E_0\big[\varphi(S_{g_t})1_{X_t>0}\big]} \underset{t\to\infty}{\longrightarrow} E_0[1_{\Lambda_s} M_s^{\varphi,0}]$$

with

$$\begin{aligned} M_s^{\varphi,0} &= \varphi(S_{g_s})X_s + \varphi(S_s)(S_s - X_s) + 1 - \Phi(S_s) \\ &= \varphi(S_{g_s})X_s^+ + \varphi(S_s)(S_s - X_s^+) + \int_{S_s}^\infty \varphi(y)dy. \end{aligned}$$

Sketch of the proof of Proposition A.8.

1) We have :

$$E_0\big[\varphi(S_{g_t})1_{X_t>0} + \psi(S_{g_t})1_{X_t<0}\big] = E_0\big[\varphi(S_{g_t}) + (\psi-\varphi)(S_t)\cdot 1_{(X_t<0)}\big]$$

because, if $X_t < 0$, then : $S_{g_t} = S_t$

$$\underset{t\to\infty}{\sim} E_0\big[\varphi(S_{g_t})\big] + E_0\big[(\psi-\varphi)(S_t)\big] \quad \text{from Lemma A.5.}$$

$$\underset{t\to\infty}{\sim} 2\cdot\sqrt{\frac{2}{\pi t}}\int_0^\infty \varphi(y)dy + \sqrt{\frac{2}{\pi t}}\int_0^\infty (\psi-\varphi)(y)dy$$

from (2.17) and [RVY, II]

$$\underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}}\int_0^\infty (\varphi+\psi)(y)dy. \tag{4.19}$$

2) We have to prove :

$$\frac{E_0\big[\varphi(S_{g_t})1_{X_t>0} + \psi(S_{g_t})1_{X_t<0}\big|\mathcal{F}_s\big]}{E_0\big[\varphi(S_{g_t})1_{X_t>0} + \psi(S_{g_t})1_{X_t<0}\big]} \underset{t\to\infty}{\longrightarrow} M_s^{\varphi,\psi}.$$

But, we have, by Lemma A.5, Theorem A.1 and (1.1), (1.2) :

$$\begin{aligned}
& E_0\big[\varphi(S_{g_t})1_{X_t>0} + \psi(S_{g_t})1_{X_t<0}\big|\mathcal{F}_s\big] \\
= \quad & E_0\big[\varphi(S_{g_t}) + (\psi-\varphi)(S_t)1_{X_t<0}\big|\mathcal{F}_s\big] \\
\underset{t\to\infty}{\sim} \quad & E_0\big[\varphi(S_{g_t})\big|\mathcal{F}_s\big] + E_0\big[(\psi-\varphi)(S_t)\big|\mathcal{F}_s\big] \\
= \quad & 2\sqrt{\frac{2}{\pi t}}\ \left[\frac{1}{2}\,\varphi(S_{g_s})|X_s| + \varphi(S_s)(S_s - X_s^+) + \int_{S_s}^{\infty}\varphi(y)dy\right] \\
& \qquad +\sqrt{\frac{2}{\pi t}}\big[(\psi-\varphi)(S_s)(S_s - X_s) + \int_{S_s}^{\infty}(\psi-\varphi)(y)dy \\
= \quad & \sqrt{\frac{2}{\pi t}}\Big[\varphi(S_{g_s})X_s + (\varphi+\psi)(S_s)(S_s - X_s) + \int_{S_s}^{\infty}(\varphi+\psi)(y)dy\Big]
\end{aligned} \tag{4.20}$$

by an easy calculation. Proposition 4.8 is then an obvious consequence of (4.20) and (4.19).

Throughout the sequel, $\Big(\Omega = \mathcal{C}\big([0,\infty[\longrightarrow \mathbb{R}\big), (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty = \underset{s}{\vee}\,\mathcal{F}_s, P_x(x\in\mathbb{R})\Big)$ denotes, as before, the canonical 1-dimensional Brownian motion and we keep the notation from the Introduction.

5 Part B.1. A local limit theorem for a class of brownian additive functionals.

B.1.1. We denote by $(L_t^x\ ;\ t\geq 0,\ x\in\mathbb{R})$ the jointly continuous family of the local times of $(X_t,\ t\geq 0)$. Let q be a positive Radon measure on $\mathbb{R}$, $q\neq 0$. Let us define :

$$A_t^q := \int_{\mathbb{R}} L_t^x q(dx) \tag{5.1}$$

$(A_t^q,\ t\geq 0)$ may be refered to as the continuous additive functional of $(X_t,\ t\geq 0)$ which is associated with q. When q admits a density with respect to Lebesgue measure, we shall preserve the former notation by still writing q for the density ; one has :

$$A_t^q = \int_0^t q(X_s)\,ds \tag{5.2}$$

from the occupation formula. Throughout the following, we shall assume that q satisfies one of the three following hypotheses :

H1. (The integrable case) $\displaystyle\int_{\mathbb{R}}\big(1+|x|\big)\,q(dx) < \infty.$

H2. (The left unilateral case) $\displaystyle\int_{-\infty}^{0}\big(1+|x|\big)\,q(dx) < \infty$ and there exists $\alpha<1$ such that $\displaystyle\varliminf_{x\to\infty} x^{2\alpha}q^{(a)}(x) \geq b > 0$ where $q^{(a)}$ denotes the absolutely continuous part of q.

H3. (The right unilateral case) $\displaystyle\int_{0}^{\infty}\big(1+|x|\big)\,q(dx) < \infty$ and there exists $\alpha<1$ such that $\displaystyle\varliminf_{x\to-\infty} |x|^{2\alpha}q^{(a)}(x) \geq b > 0.$

Of course, if the pair $\big((X_t, A_t^q),\ t \geq 0\big)$ satisfies H2 (resp. H3), then the pair $\big((-X_t, A_t^q),\ t \geq\ 0\big)$ satisfies H3 (resp. H2).
In [RVY, I], we have obtained the following result : for any $\lambda > 0$,

$$\lim_{t \to \infty} \sqrt{t}\, E_x\Big[\exp\big(-\frac{\lambda}{2} A_t^q\big)\Big] = \varphi_{\lambda q}(x) \tag{5.3}$$

(We shall indicate later precisely what is $\varphi_{\lambda q}$). The following result is a translation of the convergence result (5.3).
Theorem B.1. *Let q satisfy one of the hypotheses H1, H2 or H3, and let $(A_t^q,\, t \geq 0)$ be defined by $\big($5.1$\big)$ $\big($or (5.2)$\big)$. Then, for every $x \in \mathbb{R}$, there exists a positive, $\sigma-$ finite measure ν_x, carried by $\mathbb{R}_+$, such that :*

$$\sqrt{t}\, P_x(A_t^q \in dz) \underset{t \to \infty}{\longrightarrow} \nu_x(dz) \tag{5.4}$$

The convergence in (5.4) is understood in the following sense : for any function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ Borel, and sub-exponential i.e. : there exist two positive constants C_1 and C_2 such that :

$$0 \leq f(x) \leq C_1\, e^{-C_2 x}$$

then

$$\sqrt{t}\, E_x\big[f(A_t^q)\big] \underset{t \to \infty}{\longrightarrow} \int_{\mathbb{R}_+} f(z) \nu_x(dz)$$

The measure ν_x is characterised by :

$$\int_0^\infty e^{-\frac{\lambda}{2} y} \nu_x(dy) = \varphi_{\lambda q}(x) \tag{5.5}$$

B.1.2. Proof of Theorem B.1.
We first begin with some precisions, taken from [RVY, I], $\big($see also S. Kotani, [K]$\big)$ about $\varphi_{\lambda q}$, which was defined from (5.3) but admits at least another characterization, namely : $\varphi_{\lambda q}$ is the unique solution of the Sturm-Liouville equation :

$$\varphi''(dx) = \lambda \varphi(x) q(dx) \tag{5.6}$$

this equation being taken in the sense of Schwartz distributions, and subject to the following boundary conditions :

$$\text{Under H1. :} \qquad \varphi'(+\infty) = -\varphi'(-\infty) = \sqrt{\frac{2}{\pi}} \tag{5.7}$$

$$\text{Under H2. :} \qquad \varphi'(-\infty) = -\sqrt{\frac{2}{\pi}} \text{ and } \varphi(+\infty) = 0 \tag{5.8}$$

$$\text{Under H3. :} \qquad \varphi'(+\infty) = \sqrt{\frac{2}{\pi}} \text{ and } \varphi(-\infty) = 0 \tag{5.9}$$

Theorem B.1. is now an immediate consequence of the next lemma.
Lemma B.2. *Under either of the hypotheses H1, H2, or H3, the function : $\lambda \longrightarrow \varphi_{\lambda q}(x)$ $(\lambda >\ 0)$ is, for any real x, completely monotone, i.e., it satisfies :*

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \varphi_{\lambda q}(x) \geq 0 \tag{5.10}$$

Consequently, there exists a positive, σ-finite measure ν_x, carried by $\mathbb{R}_+$, such that :

$$\varphi_{\lambda q}(x) = \int_0^\infty e^{-\frac{\lambda}{2}z} \nu_x(dz) \tag{5.11}$$

We shall give two proofs for Lemma B.2.

B.1.3. A first proof of Lemma B.2.

We define, for every $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and every real $h \neq 0$:

$$D_h f(\lambda) := \frac{f(\lambda + h) - f(\lambda)}{h} \tag{5.12}$$

For $f(\lambda) := \exp - \dfrac{\lambda}{2} A_t^q$, we get :

$$(D_h)^n(f)(\lambda) = e^{-\frac{\lambda A_t^q}{2}} \Big(\frac{e^{-\frac{A_t^q h}{2}} - 1}{h} \Big)^n$$

and, hence for all $h \neq 0$:

$$(-1)^n (D_h)^n(f)(\lambda) \geq 0 \tag{5.13}$$

Consequently, taking the expectation of the LHS in (5.13), we obtain :

$$\sqrt{t}(-1)^n E_x\big[(D_h)^n \Big(\exp - \frac{\bullet}{2} A_t^q\Big)\big] \geq 0 \tag{5.14}$$

Hence, from (5.3) :

$$\sqrt{t}(-1)^n E_x\Big[(D_h)^n \Big(\exp - \frac{\bullet}{2} A_t^q\Big)\Big] \underset{t \to \infty}{\longrightarrow} (-1)^n (D_h)^n \big(\varphi_{\bullet q}(x)\big)$$

Thus :

$$(-1)^n (D_h)^n \big(\varphi_{\bullet q}(x)\big)(\lambda) \geq 0 \tag{5.15}$$

Letting $h \to 0$ in (5.15), and using the fact that : $D_h f \underset{h \to 0}{\longrightarrow} f'$, we get :

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \big(\varphi_{\lambda q}(x)\big) \geq 0 \tag{5.16}$$

B.1.4. A second proof of Lemma B.2.

We shall only give this second proof under the hypothesis H1 and for $x = 0$. In [RVY, I], Proposition 4.13, formula (4.43), we have given the following explicit formula for $\varphi_{\lambda q}(0)$:

$$\begin{aligned}\varphi_{\lambda q}(0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big[& Q_l^{(0)}(\exp - \lambda < Y, q^- >) \cdot Q_l^{(2)}(\exp - \lambda < Y, q^+ >) \\ & + Q_l^{(2)}(\exp - \lambda < Y, q^- >) Q_l^{(0)}(\exp - \lambda < Y, q^+ >)\Big] dl\end{aligned} \tag{5.17}$$

where, in this formula (5.17), the process $(Y_x,\, x \geq 0)$ is, under $Q_l^{(0)}$, $\big($resp. under $Q_l^{(2)}\big)$, a squared Bessel process with dimension 0, (resp. 2), starting from l, and we denote :

$$< Y, q^+ > = \int_0^\infty Y_x \;\; q(dx) \; ; \quad < Y, q^- > = \int_{-\infty}^0 Y_{-x} \;\; q(dx) \tag{5.18}$$

It is then clear, from (5.17) that : $\lambda \longrightarrow \varphi_{\lambda q}(0)$ is the Laplace transform of a positive measure, as an integral, with respect to the parameter l of the product of two Laplace transforms of positive measures (indexed by l).

We shall now give some examples for which the measure ν_x may be computed explicitly. We recall that ν_x is characterized by : $\displaystyle\int_0^\infty e^{-\frac{\lambda}{2}z}\nu_x(dz) = \varphi_{\lambda q}(x)$ where $\varphi_{\lambda q}(x)$ is given by (5.6) ... (5.9).

B.1.5. Computation of ν_x for $q(dy) = \delta_0(dy)$.

In this case, the hypothesis H1 is verified and $A_t^q = L_t$, is the local time at level 0

$$\begin{aligned}
\varphi_{\lambda q}(x) &= \sqrt{\frac{2}{\pi}}\Big(\frac{2}{\lambda} + |x|\Big) \quad \big(\text{cf [RVY, I], Ex. 4.8, p. 199-200}\big) \\
&= \int_0^\infty e^{-\frac{\lambda}{2}z}\Big(\sqrt{\frac{2}{\pi}}1_{z\geq 0}dz + \sqrt{\frac{2}{\pi}}|x|\delta_0(dz)\Big)
\end{aligned} \tag{5.19}$$

Thus :

$$\nu_x(dz) = \sqrt{\frac{2}{\pi}}1_{[0,\infty]}(z)dz + \sqrt{\frac{2}{\pi}}|x|\delta_0(dz) \tag{5.20}$$

B.1.6. Computation of ν_x for $q(dy) = \delta_a(dy) + \delta_b(dy)$ $(a < b)$.

In this case, the hypothesis H1 is verified and $A_t^q = L_t^a + L_t^b$ where $(L_t^a,\ t \geq 0)$ resp. $(L_t^b,\ t \geq 0)$ denotes the local time at level a resp. at level b. We know $\big($see [RVY, I], ex. 4.8, p. 199-200$\big)$ that

$$\varphi_{\lambda q}(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\Big(\dfrac{1}{\lambda} + x - b\Big) & \text{if } x > b \\ \sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{\lambda} & \text{if } x \in [a,b] \\ \sqrt{\dfrac{2}{\pi}}\Big(\dfrac{1}{\lambda} + a - x\Big) & \text{if } x < a \end{cases} \tag{5.21}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{\lambda}{2}z}\Big\{\frac{1}{2}\,dz + (x-b)1_{x>b}\delta_0(dz) + (a-x)1_{x<a}\delta_0(dz)\Big\}$$

hence :

$$\nu_x(dz) = \sqrt{\frac{2}{\pi}}\Big\{\frac{1}{2}\,1_{[0,\infty[}(z)dz + (x-b)^+\delta_0(dz) + (a-x)^+\delta_0(dz) \tag{5.22}$$

B.1.7. Computation of ν_x, for $q(x) = e^{2x}$.

In this case, the hypothesis H2 is satisfied and $A_t^q = \displaystyle\int_0^t e^{2X_s}ds$.

To begin with, we show :

$$\varphi_{\lambda q}(x) = \sqrt{\frac{2}{\pi}}\,K_0\,(\sqrt{\lambda}\,e^x) \tag{5.23}$$

where K_0 denotes the Bessel-Mc Donald function with index 0 $\big($see [Leb], p. 108$\big)$.

Let $\psi(x) := \sqrt{\frac{2}{\pi}}\, K_0\,(\sqrt{\lambda}\, e^x)$. To check (5.23), it suffices to see that :

$$\psi''(x) = \lambda e^{2x}\psi(x),\ \psi(x) \underset{x\to\infty}{\longrightarrow} 0,\ \psi'(x) \underset{x\to-\infty}{\longrightarrow} -\sqrt{\frac{2}{\pi}} \tag{5.24}$$

Now (5.24) follows from $\big($see [Leb], p. 110$\big)$:

$$K_0' = -K_1,\ -K_1'(z) = \frac{1}{z}\, K_1(z) + K_0(z)$$

and

$$\psi(x) \underset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\Big(\frac{\pi}{2\sqrt{\lambda}\, e^x}\Big) e^{-\sqrt{\lambda}\, e^x} \underset{x\to\infty}{\longrightarrow} 0 \qquad \big([\text{Leb}],\ p.123\big)$$

$$\psi'(x) = -\sqrt{\frac{2}{\pi}}\, \sqrt{\lambda}\, e^x K_1(\sqrt{\lambda}\, e^x) \underset{x\to-\infty}{\sim} -\sqrt{\frac{2}{\pi}}\, \sqrt{\lambda}\, e^x \frac{1}{2}\, \frac{2}{\sqrt{\lambda}\, e^x} \underset{x\to-\infty}{\longrightarrow} -\sqrt{\frac{2}{\pi}} \qquad \big([\text{Leb}],\ p.111\big)$$

This proves (5.23). But, we also have :

$$\begin{aligned} K_0(\sqrt{\lambda}\, e^x) &= \frac{1}{2}\int_0^\infty e^{-t-\frac{\lambda e^{2x}}{4t}}\, \frac{dt}{t} \qquad \big(\text{cf [Leb], p. 119}\big) \\ &= \frac{1}{2}\int_0^\infty e^{-\frac{\lambda u}{2}-\frac{e^{2x}}{2u}}\, \frac{du}{u} \end{aligned}$$

Hence :

$$\nu_x(dz) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{e^{2x}}{2z}}\, 1_{[0,\infty[}(z)\, \frac{dz}{z} \tag{5.25}$$

B.1.8. Computation of ν_x for $q_0(dx) = 1_{]-\infty,0]}(x)dx$
Here, it is the hypothesis H3 which is satisfied, and

$$A_t^{q_0} = \int_0^t 1_{]-\infty,0]}(X_s)ds$$

By scaling, one has, under P_0 : $A_t^{q_0} \overset{(law)}{=} t\, A_1^{q_0}$, and one knows that under $P_0, A_1^{q_0}$ follows the arc sine law, i.e., the beta $\Big(\frac{1}{2}, \frac{1}{2}\Big)$ law. We shall recall the law of $A_t^{q_0}$ under P_x for any $x \in \mathbb{R}$, (see Section 6, B.2), which will allow to find the following result :

$$\nu_x(dz) = x_+\sqrt{\frac{2}{\pi}}\, \delta_0(dz) + \frac{1}{\pi}\, e^{-\frac{x_-^2}{2z}}\, 1_{[0,\infty[}(z)\frac{dz}{\sqrt{z}} \tag{5.26}$$

For the moment, we shall prove (5.26) without using the explicit law of $A_t^{q_0}$. For this purpose, we already observe that :

$$\varphi_{\lambda q_0}(x) = \sqrt{\frac{2}{\pi}}\Big\{e^{x\sqrt{\lambda}}\frac{1}{\sqrt{\lambda}}\, 1_{x\le 0} + \Big(x + \frac{1}{\sqrt{\lambda}}\Big)1_{x>0}\Big\} \tag{5.27}$$

Indeed we have :

$$\varphi''_{\lambda q_0}(x) = \lambda\, 1_{]-\infty,0]}(x)\varphi_{\lambda q}(x), \qquad \varphi'_{\lambda q_0}(+\infty) = \sqrt{\frac{2}{\pi}}, \qquad \varphi_{\lambda q_0}(-\infty) = 0.$$

Then, it remains to see that :

$$\int_0^\infty e^{-\frac{\lambda}{2}z}\nu_x(dz) = \varphi_{\lambda q_0}(x) \tag{5.28}$$

where ν_x is defined via (5.26) and $\varphi_{\lambda q_0}(x)$ by (5.27). Now, for $x > 0$, one has :

$$\begin{aligned}\int_0^\infty e^{-\frac{\lambda}{2}z}\nu_x(dz) &= x_+\sqrt{\frac{2}{\pi}} + \frac{1}{\pi}\int_0^\infty e^{-\frac{\lambda z}{2}}\frac{dz}{\sqrt{z}} \\ &= x_+\sqrt{\frac{2}{\pi}} + \frac{1}{\pi}\sqrt{\frac{2}{\lambda}}\,\Gamma(1/2) = x_+\sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\lambda}} = \varphi_{\lambda q}(x)\end{aligned}$$

whereas for $x < 0$:

$$\int_0^\infty e^{-\frac{\lambda}{2}z}\nu_x(dz) = \frac{1}{\pi}\int_0^\infty e^{-\frac{\lambda}{2}z-\frac{x^2}{2z}}\frac{dz}{\sqrt{z}} = \frac{2}{\pi}K_{1/2}\big(|x|\sqrt{\lambda}\big)\Big(\frac{x^2}{\lambda}\Big)^{1/4} \qquad \big(\text{see [Leb], p. 119}\big)$$

However, one has : $K_{1/2}\big(|x|\sqrt{\lambda}\big) = \Big(\dfrac{\pi}{2|x|\sqrt{\lambda}}\Big)^{1/2} e^{-|x|\sqrt{\lambda}}$. Hence :

$$\int_0^\infty e^{-\frac{\lambda}{2}z}\nu_x(dz) = \frac{2}{\pi}\Big(\frac{x^2}{\lambda}\Big)^{1/4}\Big(\frac{\pi}{2|x|\sqrt{\lambda}}\Big)^{1/2} e^{-|x|\sqrt{\lambda}} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\lambda}}\,e^{-|x|\sqrt{\lambda}} = \varphi_{\lambda q}(x)$$

B.1.8. Computation of ν_x when $q(y) = 1_{[a,b]}(y) \quad (a < b)$.

The hypothesis (H1) is satisfied and $A_t^q = \displaystyle\int_0^t 1_{[a,b]}(X_s)\,ds$. We shall prove that :

$$\nu_x^{(a,b)}(dz) = \begin{cases} \sqrt{\frac{2}{\pi}}(x-b)\delta_0(dz) + \frac{1}{\pi\sqrt{z}}\,1_{[0,\infty[}(z)dz\Big(1 + 2\sum\limits_{n=1}^\infty e^{-\frac{n^2(b-a)^2}{2z}}\Big) & \text{if } x > b \\ \sqrt{\frac{2}{\pi}}\,(a-x)\delta_0(dz) + \frac{1}{\pi\sqrt{z}}1_{[0,\infty[}(z)dz\Big(1 + 2\sum\limits_{n=1}^\infty e^{-\frac{n^2(b-a)^2}{2z}}\Big) & \text{if } x < a \\ \frac{1}{\pi\sqrt{z}}\sum\limits_{n=0}^\infty\Big(e^{-\frac{\left(n(b-a)+b-x\right)^2}{2z}} + e^{-\frac{\left(n(b-a)+(x-a)\right)^2}{2z}}\Big)1_{[0,\infty[}(z)dz & \text{if } x \in [a,b] \end{cases} \tag{5.29}$$

Here, the explicit form of $\varphi_{\lambda q}^{(a,b)}(x)$ is $\big($see [RVY, I], Ex. 4.7, p. 199$\big)$:

$$\varphi_{\lambda q}^{(a,b)}(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\Bigg(\dfrac{1}{\sqrt{\lambda}\tanh\big(\sqrt{\lambda}\frac{b-a}{2}\big)} + x - b\Bigg) & \text{if } x > b \\ \sqrt{\frac{2}{\pi}}\Bigg(\dfrac{1}{\sqrt{\lambda}\tanh\big(\sqrt{\lambda}\frac{b-a}{2}\big)} + a - x\Bigg) & \text{if } x < a \\ \sqrt{\frac{2}{\pi}}\Bigg(\dfrac{\cosh\big(\sqrt{\lambda}\,\big(x-\frac{a+b}{2}\big)\big)}{\sqrt{\lambda}\big(\sinh\sqrt{\lambda}\frac{b-a}{2}\big)\big)}\Bigg) & \text{if } x \in [a,b] \end{cases} \tag{5.30}$$

It now remains to prove that :

$$\int_0^\infty e^{-\frac{\lambda}{2}z}\nu_x^{(a,b)}(dz) = \varphi_{\lambda q}^{(a,b)}(x) \tag{5.31}$$

where $\nu_x^{(a,b)}$ is defined via (5.29) and $\varphi_{\lambda q}^{(a,b)}$ via (5.30). But, (5.31) follows, after some elementary computations from the identities, for every real u and $v > 0$:

$$\begin{aligned}
\frac{\cosh\big(\sqrt{\lambda}u\big)}{\sqrt{\lambda}\sinh\big(\sqrt{\lambda}v\big)} &= \sum_{n=0}^\infty \int_0^\infty dh\,(e^{-\sqrt{\lambda}\big(h+(2n+1)v-u\big)} + e^{-\sqrt{\lambda}\big(h+(2n+1)v+u\big)} && (5.32)\\
&= \sum_{n=0}^\infty \int_0^\infty dh \int_0^\infty ds\,\big(H_{h+(2n+1)v-u}(s) + H_{h+(2n+1)v+u}(s)\big)e^{-\lambda s} && (5.33)
\end{aligned}$$

with

$$H_a(u) := \frac{a}{2\sqrt{\pi u^3}}\,e^{-a^2/4u} = \frac{-1}{\sqrt{\pi u}}\,\frac{\partial}{\partial a}\,(e^{-\frac{a^2}{4u}}) \qquad (a>0)$$

Passing from (5.32) to (5.33) is obtained by using the elementary formula :

$$e^{-\sqrt{\lambda}a} = \int_0^\infty e^{-\lambda u}H_a(u)du = \int_0^\infty e^{-\lambda u}\frac{a}{2\sqrt{\pi u^3}}\,e^{-\frac{a^2}{4u}}du \tag{5.34}$$

(Note that (5.34) is nothing else but a translation of : $E\big(e^{-\frac{\lambda^2}{2}T_a}\big) = \exp(-\lambda a)$, where T_a denotes the hitting time of level a by Brownian motion starting from 0).

We now show (5.32).

$$\begin{aligned}
\frac{\cosh\big(\sqrt{\lambda}u\big)}{\sqrt{\lambda}\sinh\big(\sqrt{\lambda}v\big)} &= \frac{1}{\sqrt{\lambda}}\,e^{-\sqrt{\lambda}(v-u)}\,\frac{1+e^{-2\sqrt{\lambda}u}}{1-e^{-2\sqrt{\lambda}v}}\\
&= \frac{1}{\sqrt{\lambda}}\,e^{-\sqrt{\lambda}(v-u)}(1+e^{-2\sqrt{\lambda}u})\Big(\sum_{n=0}^\infty e^{-2n\sqrt{\lambda}v}\Big)\\
&= \frac{1}{\sqrt{\lambda}}\Big\{\sum_{n=0}^\infty e^{-\sqrt{\lambda}\{v-u+2nv\}} + \sum_{n=0}^\infty e^{-\sqrt{\lambda}\big\{2(u+nv)+(v-u)\big\}}\Big\}\\
&= \frac{1}{\sqrt{\lambda}}\Big\{\sum_{n=0}^\infty \Big(e^{-\sqrt{\lambda}\big((2n+1)v-u\big)} + e^{-\sqrt{\lambda}\big(u+(2n+1)v\big)}\Big)\Big\}\\
&= \int_0^\infty e^{-\sqrt{\lambda}h}\Big\{\sum_{n=0}^\infty e^{-\sqrt{\lambda}\big((2n+1)v-u\big)} + e^{-\sqrt{\lambda}\big(u+(2n+1)v\big)}\Big\}dh\\
&= \sum_{n=0}^\infty \int_0^\infty \big(e^{-\sqrt{\lambda}\big(h+(2n+1)v-u\big)} + e^{-\sqrt{\lambda}\big(h+(2n+1)v+u\big)}\big)\,dh.
\end{aligned}$$

Remark B.3.

1. If in formula (5.29), we take : $b=0$, and we let a tend to $-\infty$, we obtain :

$$\lim_{a\to-\infty}\nu^{a,0}(dz) = \begin{cases} \sqrt{\dfrac{2}{\pi}}x_+\delta_0(dz) + \dfrac{1}{\pi\sqrt{z}}1_{[0,\infty[}(z)dz & \text{if } x>0\\ \dfrac{1}{\pi\sqrt{z}}\,e^{-\frac{x^2}{2z}}\,1_{[0,\infty[}(z)dz & \text{if } x\le 0\end{cases} \tag{5.35}$$

We note that the RHS of (5.35) is nothing else but the measure ν_x associated with $q_0(y) = 1_{]-\infty,0]}$ $\big($see (5.26)$\big)$. This may be interpreted as "a continuity property" of $\varphi^{a,b}$, as $a \longrightarrow -\infty$.
2. In the same spirit, but taking up now the example B.1.8., where we choose for q the function : $q^{(c)}(y) = \dfrac{1}{2c}\, 1_{[-c,+c]}(y)$, we have :

$$\int_0^\infty e^{-\frac{\lambda z}{2}}\, \nu_x^{(c)}\,(dz) \underset{c\downarrow 0}{\longrightarrow} \int_0^\infty e^{-\frac{\lambda z}{2}}\, \nu_x(dx) = \sqrt{\frac{2}{\pi}}\Big(\frac{2}{\lambda} + |x|\Big) \tag{5.36}$$

where ν_x is the measure associated to $q(dx) = \delta_0(dx)$ $\big($see (5.19)$\big)$. In other terms, since : $\dfrac{1}{2c}\displaystyle\int_0^t 1_{[-c,c]}(X_s)ds \underset{t\to\infty}{\longrightarrow} L_t$ *a.s.*, we witness there also a "continuity property of $\nu_x^{(c)}$ as $c \longrightarrow 0$".
Let us show (5.36) for $x = 0$; from (5.30) :

$$\int_0^\infty e^{-\frac{\lambda z}{2}}\nu_0^{(c)}(dz) = \sqrt{\frac{2}{\pi}}\left\{\frac{\cosh\big(\sqrt{\frac{\lambda}{2c}}\, c\big)}{\sqrt{\frac{\lambda}{2c}}\,\sinh\Big(\sqrt{\frac{\lambda}{2c}}\cdot c\Big)}\right\}\underset{c\downarrow 0}{\longrightarrow} \sqrt{\frac{2}{\pi}}\cdot\frac{2}{\lambda}$$

and for $x \neq 0$, and c small enough, we obtain from (5.30) that :

$$\int_0^\infty e^{-\frac{\lambda z}{2}}\nu_x^{(c)}(dz) = \sqrt{\frac{2}{\pi}}\Big\{\frac{1}{\sqrt{\frac{\lambda}{2c}}\,\tanh\Big(\sqrt{\frac{\lambda}{2c}}\cdot c\Big)} + |x-c|\Big\} \underset{c\downarrow 0}{\longrightarrow} \sqrt{\frac{2}{\pi}}\Big(\frac{2}{\lambda} + |x|\Big)$$

B.1.9. Computation of ν_x when $q(y) = 1_{[0,\infty[}(y)y^\alpha,\ \alpha > 0$.
The hypothesis H2 is satisfied, and we have : $A_t^q = \displaystyle\int_0^t \mathbf{1}_{(X_s>0)}X_s^\alpha ds.$
We now show the existence of a constant $C_\alpha > 0$ such that :

$$\nu_0(dz) = \frac{C_\alpha}{z^{\frac{1+\alpha}{2+\alpha}}}\, 1_{[0,\infty[}(z)dz \tag{5.37}$$

Indeed, thanks to the scaling property, we have :

$$\begin{aligned} E_0\big(e^{-\frac{\lambda}{2}\int_0^t 1_{X_s>0}X_s^\alpha\, ds}\big) &= E_0\big(e^{\left(-\frac{\lambda}{2}\, t^{1+\alpha/2}A_1^q\right)}\big) \\ &= E_0\Big(\exp\Big(-\frac{1}{2}\,A^q_{\lambda^{\frac{2}{2+\alpha}}t}\Big)\Big) \end{aligned} \tag{5.38}$$

Thus, multiplying (5.38) by $\sqrt{t}$ and letting t tend to $+\infty$, we obtain :

$$\varphi_{\lambda q}(0) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\,\varphi_{1\cdot q}(0) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\, c'_\alpha = c_\alpha\int_0^\infty e^{-\frac{\lambda}{2}z}\frac{dz}{z^{\frac{1+\alpha}{2+\alpha}}}$$

The same computations, performed this time with $x \neq 0$, lead to :

$$\varphi_{\lambda q}(x) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\,\varphi_{1q}\big(x\lambda^{\frac{1}{2+\alpha}}\big), \text{ i.e. } \nu_x = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\,\nu^{(\lambda)}_{x\lambda^{\frac{1}{2+\alpha}}},$$

where $\nu^{(\lambda)}$ is the image of ν by the application $z \longrightarrow \lambda z$.

Remark B.4.
We know $\big($see [R,Y], chap. X$\big)$ that, if q is an integrable function, then :

$$\frac{1}{\sqrt{t}}\int_0^t q(x+X_s)ds \xrightarrow[t\to\infty]{\text{law}} \Big(\int q(x+y)dy\Big)|N| = \Big(\int q(y)dy\Big)|N| \tag{5.39}$$

where N is a standard gaussian variable, and on the LHS of (5.39), $(X_s,\ s\geq 0)$ is a Brownian motion starting from 0. Let g denote the density of the r.v. $\bar{q}|N|$ with $\bar{q}=\displaystyle\int q(y)dy$ that is :

$$g(z)=\frac{1}{\bar{q}}\sqrt{\frac{2}{\pi}}e^{-\frac{z^2}{2(\bar{q})^2}}1_{[0,\infty[}(z)$$

Let us now consider the <u>supplementary hypothesis $\widetilde{H}$</u>, which seems reasonable enough in view of (5.39) that the density $g_t(x,\cdot)$ of the r.v. $\displaystyle\frac{1}{\sqrt{t}}\int_0^t q(x+X_s)\,ds$ converges, as $t\longrightarrow\infty$, uniformly on every compact, towards g.
However, this would imply that, for every function h, which is continuous with compact support, one has :

$$\begin{aligned}
\sqrt{t}\,E_x\big[h(A_t^q)\big] &= \sqrt{t}\,E_x\Big[h\Big(\frac{A_t^q}{\sqrt{t}}\cdot\sqrt{t}\Big)\Big] \\
&= \sqrt{t}\int_0^\infty h(z\sqrt{t})\,g_t(x,z)\,dz \\
&= \int_0^\infty h(y)g_t\Big(x,\frac{y}{\sqrt{t}}\Big)dy \underset{t\to\infty}{\longrightarrow} \int_0^\infty h(y)g(0)dy
\end{aligned}$$

But, from Theorem B.1., we know that :

$$\sqrt{t}\,E_x\big[h(A_t^q)\big]\underset{t\to\infty}{\longrightarrow}\int_{\mathbb{R}_+}h(x)\nu_x(dz)$$

Thus, this would imply that the measure $\nu_x(dz)$ would be equal to :

$$\frac{1}{\bar{q}}\sqrt{\frac{2}{\pi}}1_{[0,\infty[}dz \tag{5.40}$$

so that, the measure ν_x would not depend on x, and would be proportional to Lebesgue measure on $\mathbb{R}_+$. But, this is not the case for either of the examples B.1.4 to B.1.9. Consequently, the hypothesis $\widetilde{H}$ is not satisfied for the corresponding $q\,'s$. It would be of interest to know for which $q\,'s$, if any, it is satisfied.

B.1.10. Penalisation by $h(A_t^q)$.
Let q satisfy one of the previous hypotheses H1, H2 or H3, and denote, as before :
$A_t^q=\displaystyle\int_{\mathbb{R}}L_t^x q(dx)\Big(=\int_0^t q(X_s)ds\quad\text{if } q \text{ admits a density}\Big)$. Let now $h:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that :

$$\sqrt{t}\,E_x\big[h(A_t^q)\big]\underset{t\to\infty}{\longrightarrow}\int_0^\infty h(x)\nu_x(dz) \tag{5.41}$$

Then, (5.41) is satisfied, from Theorem B.1., as soon as h is sub-exponential (for example if h is continuous, with compact support). We shall now study the penalisation of Wiener measure by the functional $h(A_t^q)$, i.e. : we shall study the limit, as $t \longrightarrow \infty$, of :

$$\frac{E_x\big(1_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} \qquad (s \geq 0,\ \Lambda_s \in \mathcal{F}_s) \tag{5.42}$$

We have already made this study in two situations :
1) $q(dx) = \delta_0(dx)$, $A_t^q = L_t$ $\quad$ (cf [RVY, II]) ;
2) $A_t^q = \displaystyle\int_{\mathbb{R}} L_t^y\, q(dy)$, $h(u) = \exp\Big(-\dfrac{\lambda}{2}\,u\Big)$ $\quad$ (cf [RVY, I]).
This time, Theorem B.1. allows us to obtain :
Theorem B.5. : *Let q, A^q and h as above. Then :*
1) *For every $s \geq 0$, and every $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} \frac{E_x\big(1_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} \quad \text{exists} \tag{5.43}$$

2) *This limit equals $E_x(\mathbf{1}_{\Lambda_s} M_s^{h,q}) := Q^{h,q}(\Lambda_s)$, where*

$$M_s^{h,q} := \frac{\displaystyle\int_{\mathbb{R}_+} \nu_{X_s}(dz)\, h(z + A_s^q)}{\displaystyle\int_{\mathbb{R}_+} \nu_x(dz)\, h(z)} \tag{5.44}$$

Furthermore,$(M_s^{h,q}, s \geq 0)$ is a positive martingale. In the case when $h(u) := e^{-\frac{\lambda}{2}u}$ $(u, \lambda \geq 0)$, we then obtain :

$$M_s^{h,q} = \frac{\varphi_{\lambda q}(X_s)}{\varphi_{\lambda q}(x)} \exp\Big(-\frac{\lambda}{2}\, A_s^q\Big) \tag{5.45}$$

Proof of Theorem B.5.
We have :

$$\frac{E_x\big(\mathbf{1}_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} = \frac{E_x\big(\mathbf{1}_{\Lambda_s} E_b\big(h(a + A_{t-s}^q)\big)}{E_x\big(h(A_t^q)\big)}$$

from the Markov property, where $b = X_s$ and $a = A_s^q$. Thus, from Theorem B.1. :

$$E_x\big(h(A_t^q)\big) \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}} \int_0^\infty \nu_x(dz) h(z)$$

$$\text{and} \qquad E_b\big(h(a + A_{t-s}^q)\big) \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t-s}} \int_0^\infty \nu_b(dz) h(a+z)$$

Hence :

$$\begin{aligned} \frac{E_x\big(\mathbf{1}_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} &\underset{t\to\infty}{\sim} \frac{\sqrt{t}}{\sqrt{t-s}}\ \frac{E_x\big(\mathbf{1}_{\Lambda_s} \int_{\mathbb{R}_+} \nu_{X_s}(dz) h(z + A_s^q)\big)}{\int_{\mathbb{R}_+} h(z) \nu_x(dz)} \\ &\underset{t\to\infty}{\longrightarrow} E_x(\mathbf{1}_{\Lambda_s} M_s^{h,q}) \end{aligned}$$

In the preceeding lines, we have been a little careless concerning the exchange of limit and expectation. Likewise, although it is easy to see that $(M_s^{h,q},\ s \geq 0)$ is a local martingale, some care is needed in order to show that it is a true martingale. However, all this is correct as soon as h is sub-exponential.

6 Part B.2. A detailed study for $q_0 = 1_{]-\infty,0]}$, $A_t^- := \int_0^t 1_{(X_s<0)} ds$.

Throughout this section, we choose $q_0 = 1_{]-\infty,0]}$. Thus, the hypothesis H3 is now satisfied. We shall study this situation in detail, which we are able to do as we know $\big($see [Y]$\big)$ the law of $A_t^{q_0} = \displaystyle\int_0^t q_0(X_s)ds$ under P_x, for every real x $\big($see (6.5) and (6.7) below$\big)$. We shall, successively :

- compute explicitly the measure ν_x starting from the knowledge of the law of $A_t^{q_0}$ and we shall recover the result of point (B.1.8.) ;
- study the penalisation, not only of the process $(X_t,\, t \geq 0)$ by $h(A_t^{q_0})$, but also the penalisation of the "long bridges" by this functional ;
- describe precisely the behavior of the canonical process under the probability Q^{h,q_0}, where Q^{h,q_0} is defined via :

$$Q^{h,q_0}(\Lambda_s) = E(1_{\Lambda_s} M_s^{h,q_0}) \qquad (s \geq 0,\ \Lambda_s \in \mathcal{F}_s) \tag{6.1}$$

B.2.1. The law of A_t^- and the computation of ν_x.
To simplify notation, we denote :

$$A_t^- = \int_0^t 1_{(X_s<0)}\, ds = \int_0^t q_0(X_s) ds \tag{6.2}$$

We recall the following result, which is found in [Y]. For any $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, Borel, bounded, such that $f(x) \underset{x\to\infty}{\longrightarrow} 0$ and any $y > 0$:

$$E_0\Big[f\Big(\int_0^1 1_{X_s<y} ds\Big)\Big] = \int_0^1 \frac{du}{\pi\sqrt{u(1-u)}}\, e^{-\frac{y^2}{2u}} f(u) + f(1)\sqrt{\frac{2}{\pi}} \int_0^y e^{-\frac{\alpha^2}{2}} d\alpha \tag{6.3}$$

whereas, for any $y < 0$, we use :

$$\int_0^1 1_{(X_s<y)}\, ds \overset{\text{law}}{=} \int_0^1 1_{(X_s>-y)} ds \overset{\text{law}}{=} 1 - \int_0^1 1_{(X_s<-y)} ds \tag{6.4}$$

and by the scaling property :

$$\begin{aligned} & E_x\Big[f\Big(\int_0^t 1_{(X_s<0)} ds\Big)\Big] \\ = \;& E_x\big(f(A_t^-)\big) = E_0\Big(f\Big(\int_0^t 1_{(X_s<-x)} ds\Big)\Big) = E_0\Big(f\Big(t\int_0^1 1_{\left(X_s<-\frac{x}{\sqrt{t}}\right)} ds\Big)\Big) \end{aligned}$$

hence, from (6.3) and (6.4), if $x \leq 0$:

$$E_x\big[f(A_t^-)\big] = \int_0^t \frac{dv}{\pi\sqrt{v(t-v)}}\, e^{-\frac{x^2}{2v}} f(v) + f(t)\sqrt{\frac{2}{\pi}} \int_0^{\frac{|x|}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}} d\alpha \tag{6.5}$$

$$\underset{t\to\infty}{\sim} \frac{1}{\pi\sqrt{t}} \int_0^\infty \frac{dv}{\sqrt{v}}\, e^{-\frac{x^2}{2v}} f(v) \tag{6.6}$$

whereas, if $x > 0$:

$$E_x\big[f(A_t^-)\big] = f(0)\sqrt{\frac{2}{\pi}}\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}}\,d\alpha + \int_0^t \frac{dv}{\pi\sqrt{v(t-v)}}\, e^{-\frac{x^2}{2(t-v)}} f(v) \tag{6.7}$$

$$\underset{t\to\infty}{\sim} f(0)\sqrt{\frac{2}{\pi}}\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}}\,d\alpha + \frac{1}{\pi\sqrt{t}}\int_0^\infty \frac{dv}{\sqrt{v}}\, f(v) \tag{6.8}$$

Thus, we obtain :

$$\sqrt{t}\, E_x\big[f(A_t^-\big] \underset{t\to\infty}{\longrightarrow} \int_0^\infty f(z)\nu_x(dz)$$

with

$$\nu_x(dz) = x_+\sqrt{\frac{2}{\pi}}\delta_0(dz) + \frac{1}{\pi}\, e^{-\frac{x_-^2}{2z}}\, 1_{[0,\infty]}(z)\frac{dz}{\sqrt{z}}$$

which is precisely (5.26).

B.2.2. Penalisation by $h(A_t^-)$. A study of "long bridges" and of the Q^h−process.
We recall that, from (6.5), the density of A_t^- under P_0, which we denote by $p_{A_t^-}$, equals :

$$p_{A_t^-}(y) = \frac{1}{\pi}\,\frac{1}{\sqrt{y(t-y)}}\,\mathbf{1}_{[0,t]}(y) \qquad \text{(: the arc sine law).} \tag{6.9}$$

Throughout the following, h denotes a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that :

$$\int_0^\infty \frac{dy}{\sqrt{y}}\, h(y) < \infty$$

and we assume, without loss of generality, that :

$$\int_0^\infty \frac{dy}{\sqrt{y}}\, h(y) = 1 \tag{6.10}$$

Theorem B.6.
1) *For every $s \geq 0$ and every $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} E_0(1_{\Lambda_s}|A_t^- = a) = Q^{(a)}(\Lambda_s) \tag{6.11}$$

with

$$\begin{aligned} Q^{(a)}(\Lambda_s) &:= \sqrt{\frac{2}{\pi}}\,\frac{1_{a<s}}{\sqrt{s-a}}\, E_0(1_{\Lambda_s}X_s^+|A_s^- = a) \\ &+ E\Big[1_{\Lambda_s}\sqrt{\frac{a}{a-A_s^-}}\, 1_{(A_s^-<a)}\, e^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\Big] \end{aligned} \tag{6.12}$$

$\Big($*recall that $X_s^+ = 0 \vee X_s,\ X_s^- = -(X_s \wedge 0)$ and $A_s^- = \int_0^s 1_{X_u<0}\, du$*$\Big)$
2) *For every function h which satisfies (6.10), for every $s \geq 0$ and any $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} \frac{E_0\big(1_{\Lambda_s}h(A_t^-)\big)}{E_0\big(h(A_t^-)\big)} = E(1_{\Lambda_s}M_s^h) \tag{6.13}$$

where $(M_s^h,\, s\geq 0)$ *is the positive martingale given by :*

$$M_s^h := \sqrt{2\pi}\, X_s^+\, h(A_s^-) + \int_0^\infty \frac{dy}{\sqrt{y}}\, e^{-\frac{(X_s^-)^2}{2y}}\, h(A_s^- + y) \tag{6.14}$$

(note that $M_0^h = 1$*).*

3) *Formula (6.13) induces a probability* Q^h *on* $(\Omega, \mathcal{F}_\infty)$*, which admits the following disintegration :*

$$Q^h(\Lambda) = \int_0^\infty Q^{(a)}(\Lambda)\, \frac{h(a)}{\sqrt{a}}\, da \qquad (\Lambda \in \mathcal{F}_\infty) \tag{6.15}$$

where $Q^{(a)}$ *is given by (6.12).*

4) *Under* Q^h*, the canonical process* $\big(\Omega, (X_t,\, t\geq 0)\big)$ *satisfies :*

$$i) \quad A_\infty^- \text{ is finite a.s., and admits as density } \frac{h(y)}{\sqrt{y}}\, 1_{y>0} \tag{6.16}$$

$$ii) \quad \text{let } g = \inf\{t\ ;\ A_t = A_\infty\} = \sup\{t\ ;\ X_t \leq 0\} \tag{6.17}$$

Then $Q^h(g<\infty) = 1$

iii) the processes $(X_t,\, t\leq g)$ and $(X_{g+t},\, t\geq 0)$ *are independent ;*

iv) the process $(X_{g+t},\ t\geq 0)$ *is a 3-dimensional Bessel process starting from 0.*

Moreover, while proving Theorem B.6., we shall give a precise description of the process $(X_t\ ;\ t\leq g)$.

Proof of Theorem B.6.

B.2.3. We prove point 1) of Theorem B.6.

For this purpose, we choose a function h, which is Borel, positive, and satisfies (6.10).
We first write :

$$E_0\big(1_{\Lambda_s} h(A_t^-)\big) = \int_0^t E_0(1_{\Lambda_s} | A_t^- = a)\, p_{A_t^-}(a) h(a) da \tag{6.18}$$

then, conditioning with respect to $\mathcal{F}_s$, we obtain :

$$E_0\big(1_{\Lambda_s} h(A_t^-)\big) = E_0\Big(1_{\Lambda_s}\, E_0\Big(h\Big(a + \int_0^{t-s} 1_{(X_u<-x)}\, du\Big)\Big)\Big) \tag{6.19}$$

with $a = A_s^-$ and $x = X_s$. Using now (6.6) and (6.7), we obtain :

$$\begin{aligned} E_0\big(1_{\Lambda_s} h(A_t)\big) &= E_0\Big(1_{\Lambda_s} 1_{x<0}\Big(\Big(\int_0^{t-s} \frac{dv}{\pi\sqrt{v(t-s-v)}}\, e^{-\frac{x^2}{2v}} h(a+v)\Big) + h(a+t-s)\psi\Big(\frac{|x|}{\sqrt{t-s}}\Big)\Big)\Big) \\ &\quad + E_0\Big(1_{\Lambda_s} 1_{x>0}\Big[h(a)\psi\Big(\frac{x}{\sqrt{t-s}}\Big) + \int_0^{t-s} \frac{dv}{\pi\sqrt{v(t-s-v)}}\, e^{-\frac{x^2}{2(t-s-v)}} h(a+v)\Big]\Big) \quad (6.20) \\ &:= (1)_t + (2)_t \quad (6.21) \end{aligned}$$

where $\psi\Big(\dfrac{x}{\sqrt{t}}\Big) := P\Big(|N| \leq \dfrac{x}{\sqrt{t}}\Big) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\alpha^2}{2}} d\alpha \underset{t\to\infty}{\sim} \sqrt{\dfrac{2}{\pi}}\, \dfrac{x}{\sqrt{t}}$.

We now study successively $(1)_t$ and $(2)_t$. We rewrite $(1)_t$ in the form :

$$
\begin{aligned}
&(1)_t \\
&= \int_0^s p_{A_s^-}(a)da\, E_0\Big(1_{\Lambda_s}1_{X_s<0}\Big(\int_0^{t-s}\frac{dv}{\pi\sqrt{v(t-s-v)}}\,e^{-\frac{X_s^2}{2v}}h(a+v) \\
&\qquad\qquad + h(a+t-s)\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\Big|A_s^-=a\Big) \\
&= \int_0^{t-s}\frac{dv}{\pi\sqrt{v(t-s-v)}}\int_v^{v+s} da\, p_{A_s^-}(a-v)\,E_0\big(1_{\Lambda_s}1_{X_s<0}\,e^{-\frac{X_s^2}{2v}}|X_s=a-v\big)h(a) \\
&\quad + \int_t^{s+t} p_{A_s^-}(a-t)E_0\Big(1_{\Lambda_s}1_{X_s<0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\Big|A_s^-=a-t\Big)h(a)da \qquad (6.22)
\end{aligned}
$$

Similarly :

$$
\begin{aligned}
&(2)_t \\
&= \int_0^s p_{A_s^-}(a)\,E_0\Big(1_{\Lambda_s}1_{X_s>0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\Big|A_s^-=a\Big)h(a)da \qquad (6.23) \\
&\quad + \int_0^s da\, p_{A_s^-}(a)\int_0^{t-s}\frac{dv}{\pi\sqrt{v(t-s-v)}}E_0\big(1_{\Lambda_s}1_{X_s>0}\,e^{-\frac{X_s^2}{2(t-s-v)}}\big|A_s^-=a\big)h(a+v) \\
&= \int_0^s p_{A_s^-}(a)\,E_0\Big(1_{\Lambda_s}1_{X_s>0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\big|A_s^-=a\Big)h(a)da \qquad (6.24) \\
&\quad + \int_0^{t-s}\frac{dv}{\pi\sqrt{v(t-s-v)}}\int_v^{v+s} p_{A_s^-}(a-v)\,E_0\Big(1_{\Lambda_s}1_{X_s>0}\,e^{-\frac{X_s^2}{2(t-s-v)}}\big|A_s^-=a-v\Big)h(a)da
\end{aligned}
$$

Then, comparing (6.18), (6.22), and (6.24), it follows that :

$$E_0\big(1_{\Lambda_s}|A_t^-=a\big) = (\tilde{1})_t + (\tilde{2})_t$$

with :

$$
\begin{aligned}
&(\tilde{1})_t \\
&= \frac{1}{p_{A_t^-}(a)}\int_0^{t-s}\frac{dv}{\pi\sqrt{v(t-s-v)}}\,1_{v<a<v+s}\,p_{A_s^-}(a-v)\,E_0\Big(1_{\Lambda_s}1_{X_s<0}\,e^{-\frac{X_s^2}{2v}}\big|A_s^-=a-v\Big) \\
&\quad + \frac{1}{p_{A_t^-}(a)}\,1_{t<a<s+t}\,p_{A_s^-}(a-t)\,E_0\Big(1_{\Lambda_s}1_{X_s<0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\big|A_s^-=a-t\Big) \\
&\underset{t\to\infty}{\longrightarrow} \int_0^s \frac{\sqrt{a}}{\sqrt{a-w}}\,p_{A_s^-}(w)\,E_0\big(1_{\Lambda_s}1_{X_s<0}\,e^{-\frac{X_s^2}{2(a-w)}}\big|A_s^-=w\big)dw \\
&= E_0\Big(1_{\Lambda_s}1_{X_s<0}\,e^{-\frac{X_s^2}{2(a-A_s^-)}}\,\frac{\sqrt{a}}{\sqrt{a-A_s^-}}\,1_{A_s^-<a}\Big) \qquad (6.25)
\end{aligned}
$$

since $p_{A_t^-}(a) = \dfrac{1}{\pi\sqrt{a(t-a)}}\, 1_{[0,t]}(a)$. Similarly, one has :

$$
\begin{aligned}
&(\tilde{2})_t \\
&= \frac{p_{A_s^-}(a)}{p_{A_t^-}(a)}\, 1_{a<s}\, E_0\Big(1_{\Lambda_s} 1_{X_s>0}\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\big|A_s^- = a\Big) \\
&\quad +1_{v<a<v+s}\cdot \frac{1}{p_{A_t^-}(a)}\int_0^{t-s}\frac{dv}{\pi\sqrt{v(t-s-v)}}\, p_{A_s^-}(a-v)\, E_0\Big(1_{\Lambda_s}1_{X_s>0}e^{-\frac{X_s^2}{2(t-s-v)}}\big|A_s^- = a-v\Big) \\
&\underset{t\to\infty}{\longrightarrow} \frac{1_{a<s}}{\sqrt{s-a}}\sqrt{\frac{2}{\pi}}\, E_0\Big(1_{\Lambda_s}X_s^+\big|A_s^- = a\Big) + E_0\Big(1_{\Lambda_s}1_{X_s>0}\sqrt{\frac{a}{a-A_s^-}}\, 1_{A_s^-<a}\Big)
\end{aligned}
$$

Hence, point 1 of Theorem B.6. follows.

B.2.4. We now prove point 2 of Theorem B.6.

In fact, this point has already been shown while proving Theorem B.5. With the help of the form of M^h as given by (5.43) and the explicit computation of ν_x $\big($see formula (5.26)$\big)$, we obtain :

$$
\begin{aligned}
M_s^h &= \frac{\displaystyle\int_0^\infty \nu_{X_s}(dy)\, h(A_s^-+y)}{\displaystyle\int_0^\infty \nu_0(dy)\, h(y)} = \frac{\displaystyle\int_0^\infty h(A_s^-+y)\Big[X_s^+\sqrt{\frac{2}{\pi}}\delta_0(dy) + \frac{1}{\pi}\, e^{-\frac{(X_s^-)^2}{2y}}\frac{dy}{\sqrt{y}}\Big]}{\displaystyle\frac{1}{\pi}\int_0^\infty h(y)\,\frac{dy}{\sqrt{y}}} \\
&= \sqrt{2\pi}\, X_s^+\, h(A_s^-) + \int_0^\infty \frac{dy}{\sqrt{y}}\, e^{-\frac{(X_s^-)^2}{2y}}\, h(A_s^-+y) \qquad (6.26)
\end{aligned}
$$

Now, clearly, this point 1 of Theorem B.6. which we just proved implies also point 2 of the same Theorem B.6. Indeed, we have :

$$
\frac{E_0\big(1_{\Lambda_s}h(A_t^-)\big)}{E_0\big(h(A_t^-)\big)} = \frac{\displaystyle\int_0^t E_0\big(1_{\Lambda_s}|A_t^- = a\big)\, h(a)\, p_{A_t^-}(a)da}{\displaystyle\int_0^t h(a)\, p_{A_t^-}(a)da}
$$

From the above point 1, and with the help of the explicit form of $p_{A_t^-}(a)$ as given by (6.9) the above quantity converges, as $t\to\infty$, towards :

$$
\frac{\displaystyle\int_0^\infty \frac{da}{\sqrt{a}}\, Q^{(a)}(\Lambda_s)h(a)}{\displaystyle\int_0^\infty \frac{h(a)da}{\sqrt{a}}} = \int_0^\infty \frac{h(a)}{\sqrt{a}}\, Q^{(a)}(\Lambda_s)da \qquad (6.27)
$$

since we assumed : $\displaystyle\int_0^\infty \frac{h(a)da}{\sqrt{a}} = 1$.

<u>It now remains to compute</u> :

$$
\begin{aligned}
\int_0^\infty \frac{h(a)}{\sqrt{a}}\, Q^{(a)}(\Lambda_s)da &= \sqrt{\frac{2}{\pi}}\int_0^s \frac{1}{\sqrt{a(s-a)}}\, E_0\big(1_{\Lambda_s}X_s^+|A_s^- = a\big)h(a)da \\
&+ \int_0^\infty \frac{h(a)}{\sqrt{a}}\, E_0\Big(1_{\Lambda_s}\sqrt{\frac{a}{a-A_s^-}}\, 1_{A_s^-<a}\, e^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\Big)da
\end{aligned}
$$

$\big($from (6.12)$\big)$

$$
\begin{aligned}
&= \sqrt{2\pi}\int_0^s p_{A_s^-}(a)\, E_0\big(1_{\Lambda_s} X_s^+ | A_s^- = a\big)\, h(a)da \\
&\quad + \int_0^\infty \frac{dy}{\sqrt{y}}\, E_0\Big(1_{\Lambda_s} e^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y)\Big) \\
&\quad \big(\text{from (6.9)}\big) \text{and after the change of variable } a - A_s^- = y \\
&= \sqrt{2\pi}\, E_0\Big(1_{\Lambda_s} X_s^+\, h(A_s^-)\Big) + \int_0^\infty E_0\Big(1_{\Lambda_s} e^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y)\Big) \frac{dy}{\sqrt{y}} \\
&= E_0(1_{\Lambda_s} M_s^h) \qquad \big(\text{from (6.14)}\big)
\end{aligned}
\tag{6.28}
$$

We now remark that point 3 in Theorem B.6. states precisely formula (6.28) we just established.

B.2.5. We now prove point 4 of Theorem B.6.

a) From formula (6.15) and from Doob's optional sampling theorem, we deduce :

$$
Q^h(A_\infty^- > a) = E[M_{\tau_a}^h], \quad \text{with } \tau_a := \inf\{t \;;\; A_t^- > a\}
\tag{6.29}
$$

But :

$$
\begin{aligned}
M_{\tau_a}^h &= \sqrt{2\pi}\, h(a)\, X_{\tau_a}^+ + \int_0^\infty \frac{dy}{\sqrt{y}}\, e^{-\frac{(X_{\tau_a}^-)^2}{2y}} h(a+y) \\
&= \int_0^\infty \frac{dy}{\sqrt{y}}\, e^{-\frac{(X_{\tau_a}^-)^2}{2y}} h(a+y) \qquad \text{since } X_{\tau_a}^+ = 0
\end{aligned}
$$

We recall that the <u>process</u> $(X_{\tau_a}^- \;;\; a \geq 0)$ is distributed as the reflecting Brownian motion $\big(|X_a|,\ a \geq 0\big)$, where $(X_a,\ a > 0)$ is a standard Brownian motion starting from 0 $\big($see, e.g., [K.S], Th. 3.1, p. 419$\big)$. Hence, we obtain :

$$
\begin{aligned}
E[M_{\tau_a}^h] &= \int_0^\infty \frac{dy}{\sqrt{y}}\, h(a+y)\, E(e^{-\frac{X_a^2}{2y}}) \\
&= \int_0^\infty \frac{dy}{\sqrt{y}}\, h(a+y) \sqrt{\frac{y}{y+a}} = \int_0^\infty \frac{dy}{\sqrt{y+a}}\, h(a+y) = \int_a^\infty \frac{dy}{\sqrt{y}}\, h(y)
\end{aligned}
$$

b) We now remark that it is easy to recover the law of A_∞^- under Q^h from points 1 and 2 in Theorem B.6. We may already prove that, under $Q^{(a)}$, one has $A_\infty^- = a$ a.s. Indeed, this follows from :

$$
\begin{aligned}
\text{if } b > a, \qquad Q^{(a)}(A_s > b) &= \sqrt{\frac{2}{\pi}}\, \frac{1_{a<s}}{\sqrt{s-a}}\, E_0\big(X_s^+ 1_{A_s^- > b} | A_s^- = a\big) \\
&\quad + E_0\Big(\sqrt{\frac{a}{a - A_s^-}}\, 1_{b<A_s<a}\, e^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\Big) = 0
\end{aligned}
$$

Hence, passing to the limit as $s \to \infty$, if $b > a$: $Q^a(A_\infty^- > b) = 0$

On the other hand, it is clear that $E_0\big(1_{A_s^- \leq a} | A_t^- = a\big) = 1 \quad (t > s)$, hence, passing to the limit as $t \to \infty$, and then, letting $s \to \infty$ we obtain :

$$
Q^{(a)}(A_\infty^- \leq a) = 1
$$

Finally, from (6.15), we get :

$$Q^h(A^-_\infty \le b) = \int_0^\infty \frac{h(a)}{\sqrt{a}}\, Q^{(a)}(A^-_\infty \le b)da = \int_0^b \frac{h(a)}{\sqrt{a}}\, da.$$

B.2.6. Computation of Azéma's supermartingale $Z_t := Q^h(g > t|\mathcal{F}_t)$.
Let

$$g = \inf\{t \ge 0\ ;\ A^-_t = A^-_\infty\} = \sup\{t \ge 0\ ;\ X_t \le 0\} \tag{6.30}$$

Lemma B.7. *The following explicit formula holds :*

$$Z_t := Q^{(h)}\big(g > t|\mathcal{F}_t\big) = 1_{(X_t<0)} + 1_{(X_t>0)}\,\frac{\displaystyle\int_0^\infty \frac{dv}{\sqrt{v}}\, h(A^-_t+v)}{M^h_t} \tag{6.31}$$

Proof of Lemma B.7. We note that, for $\Lambda_t \in \mathcal{F}_t$:

$$Q^h(1_{g>t}1_{\Lambda_t}) = Q^h(1_{\Lambda_t}1_{X_t<0}) + Q^h(1_{\Lambda_t}1_{X_t>0}1_{d_t<\infty})$$

$\big($where d_t denotes the first return time to 0 after time $t\big)$

$$= Q^h(1_{\Lambda_t}1_{X_t<0}) + E(1_{\Lambda_t}1_{X_{t>0}}M^h_{d_t})\big)$$

We have :

$$M^h_{d_t} = \sqrt{2\pi}\, h(A^-_{d_t})X^+_{d_t} + \int_0^\infty \frac{dv}{\sqrt{v}}\, h(A^-_{d_t}+v) = \int_0^\infty \frac{dv}{\sqrt{v}}\, h(A^-_t+v) \quad \big(\text{from } (6.14)\big)$$

since $X_{d_t} = 0$ and $A^-_{d_t} = A^-_t$ on the set $X_t > 0$.
Hence :

$$Q^h(1_{g>t}1_{\Lambda_t}) = Q^h\left(1_{\Lambda_t}\left(1_{X_t<0} + 1_{X_t>0}\,\frac{\displaystyle\int_0^\infty \frac{dw}{\sqrt{w}}\, h(A^-_t+w)}{M^h_t}\right)\right)$$

This proves (6.31). $\square$

B.2.7. We now prove that $Q^h(g < \infty) = 1$.
We deduce from (6.31) that :

$$\begin{aligned}
Q[g<t] &= 1 - Q[g>t] \\
&= 1 - E\Big[1_{X_t<0}M^h_t + 1_{X_t>0}\int_0^\infty \frac{dv}{\sqrt{v}}\, h(A^-_t+v)\Big] \\
&= \sqrt{2\pi}\, E\big[X^+_t h(A^-_t)\big]
\end{aligned}$$

$\big($from (6.26) and since $(M^h_t,\, t\ge 0)$ is a martingale s.t. $E(M^h_t) = 1\big)$

$$\begin{aligned}
&= \frac{\sqrt{2\pi}}{2}\, E\Big(\int_0^t h(A^-_s)\, d\,L_s\Big) \quad \text{from Itô-Tanaka formula} \\
&\underset{t\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{2}}E\Big(\int_0^\infty h(A^-_s)\, d\,L_s\Big) = \sqrt{\frac{\pi}{2}}E\Big(\int_0^\infty h(a)\, dL_{\tau_a}\Big)
\end{aligned}$$

where $(\tau_a,\ a \geq 0)$ denotes the right continuous inverse of (A_t^-)

$$= 2\sqrt{\frac{\pi}{2}}\, E\Big(\int_0^\infty h(a)\, dL_a\Big) \qquad \Big(\text{since } (X_{\tau_a}^-,\ a \geq 0) \text{ is distributed as } \big(|X_a|,\ a \geq 0\big)\Big)$$

$$= 2\sqrt{\frac{\pi}{2}} \int_0^\infty h(a)\, E(dL_a) = \int_0^\infty \frac{h(a)}{\sqrt{a}}\, da = 1$$

since $E(L_a) = \sqrt{a} \cdot \sqrt{\dfrac{2}{\pi}}$.

B.2.8. We now describe the canonical process $(X_t,\ t \geq 0)$ under Q^h.

For this purpose, we shall use the technique of enlargement of filtrations. Thus, let $(\mathcal{G}_t,\ t \geq 0)$ denote the smallest filtration which makes g a $(\mathcal{G}_t\ t \geq 0)$ stopping time, and which contains $(\mathcal{F}_t,\ t \geq 0)$.

The application of Girsanov's Theorem and (6.14) imply the existence of a $(\mathcal{F}_t,\ Q^h)$ Brownian motion $(\beta_t,\ t \geq 0)$ such that, under Q^h :

$$X_t = \beta_t + \int_0^t \frac{1}{M_s^h}\Big\{\sqrt{2\pi}\, h(A_s^-)\, 1_{X_s>0} - \Big(\int_0^\infty \frac{dw}{w^{3/2}}\, e^{-\frac{(X_s^-)^2}{2w}}\, h(A_s^- + w)\Big)\, X_s^-\Big\} ds \tag{6.32}$$

We now apply the enlargement formulae $\big($cf [J], [JY], [MY]$\big)$. We first observe that :

$$dZ_t = -\frac{\displaystyle\int_0^\infty \frac{dw}{\sqrt{w}}\, h(A_t^- + w)}{(M_t^h)^2}\, \big(\sqrt{2\pi}\; h(A_t^-) 1_{X_t>0}\, dX_t\big) + d\ \ (\text{b.v. term}) \tag{6.33}$$

where b.v. means bounded variation and therefore :

$$d < Z,\, X >_t = -\frac{\displaystyle\int_0^\infty \frac{dw}{\sqrt{w}}\, h(A_t^- + w)}{(M_t^h)^2}\, \sqrt{2\pi}\, h(A_t^-)\, 1_{(X_t>0)} dt \tag{6.34}$$

Thus, there exists a $\big((\mathcal{G}_t,\ t \geq 0),\ Q^h\big)$ Brownian motion $(\widetilde{\beta}_t,\ t \geq 0)$ such that :

$$\begin{aligned}
dX_t \;=\; & d\widetilde{\beta}_t + \frac{1}{M_t^h}\Big\{\sqrt{2\pi}\, h(A_t^-)\, 1_{X_t>0} - \Big(\int_0^\infty \frac{dw}{w^{3/2}}\, e^{-\frac{(X_t^-)^2}{2w}}\, h(A_t^- + w)\Big)\, X_t^-\Big\} dt \\
& + 1_{t<g}\Big[-\frac{\int_0^\infty \frac{dw}{\sqrt{w}}\, h(A_t^- + w)}{(M_t^h)^2}\big(\sqrt{2\pi}\, h(A_t^-)\, 1_{X_t>0}\big) \cdot \frac{M_t^h}{\int_0^\infty \frac{dw}{\sqrt{w}}\, h(A_t^- + w)}\Big] dt \\
& - 1_{t>g}\Big[-\frac{\int_0^\infty \frac{dw}{\sqrt{w}}\, h(A_t^- + w)}{(M_t^h)^2}\big(\sqrt{2\pi}\, h(A_t^-)\, 1_{X_t>0}\big) \cdot \frac{M_t^h}{\sqrt{2\pi}\, h(A_t^-)\, X_t^+}\Big] dt
\end{aligned}$$

This yields, after some simplifications :

$$X_t = \widetilde{\beta}_t - \int_0^{t\wedge g} \frac{1}{M_s^h}\Big(\int_0^\infty \frac{dw}{w^{3/2}}\, e^{-\frac{(X_s^-)^2}{2w}} h(A_s^- + w)\Big)\, X_s^-\, ds + \int_{t\wedge g}^t \frac{ds}{X_s} \tag{6.35}$$

since, after g, $X_t^- = 0$, hence $X_t^+ = X_t$.

Points 4 iii) and iv) now follow immediately from (6.35).

Remark B.2. When $h(x) = e^{-\frac{\lambda x}{2}} \quad (\lambda > 0,\ x \geq 0)$, the equation (6.35) simplifies as :

$$X_t = \widetilde{\beta}_t - \int_0^{t \wedge g} \frac{\sqrt{X_s^-}\,\lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{4}}\sqrt{2\pi}\,X_s^+ - \sqrt{X_s^-}}\,ds + \int_{t \wedge g}^t \frac{ds}{X_s} \tag{6.36}$$

and this formula (6.36) follows from :

$$\int_0^\infty \frac{dw}{w^{3/2}}\,e^{-\frac{(X_s^-)^2}{2w} - \frac{\lambda}{2}w}\,dw = \Big(\frac{(X_s^-)^2}{\lambda}\Big)^{-1/4} 2\,K_{-1/2}(\sqrt{\lambda}X_s^-)$$
$$\int_0^\infty \frac{dw}{w^{1/2}}e^{-\frac{(X_s^-)^2}{2w} - \frac{\lambda}{2}w}\,dw = \Big(\frac{(X_s^-)^2}{\lambda}\Big)^{+1/4} 2\,K_{1/2}(\sqrt{\lambda}X_s^-)$$

and from : $K_{-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) = \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}$ $\big($[Leb], p. 112 and p. 119$\big)$.

B.2.9. A little mystery

Theorem B.6. shows that the process $(X_t,\ t \geq 0)$ is not Markovian under Q^h, whereas the 2-dimensional process $\big((X_t, A_t^-),\ t \geq 0\big)$ is Markovian.

Indeed, g is not a $(\mathcal{F}_t,\ t \geq 0)$ stopping time and the dynamics of (X_t) is not the same before and after g. On the other hand, we know $\big($see [RVY, I]$\big)$ that if $h(x) := e^{-\frac{\lambda}{2}x} \quad (\lambda, x \geq 0)$, then the Q^h-process is Markovian. It is the diffusion with infinitesimal generator :

$$L^h f(x) = \frac{1}{2}\,f''(x) + \frac{\varphi'}{\varphi}\,(x) f'(x), \qquad f \in C_b^2$$

where φ denotes the unique solution of $\varphi'' = \lambda\varphi, \varphi(-\infty) = 0$; $\varphi'(+\infty) = \sqrt{\frac{2}{\pi}}\cdot$ In this case, the solution of this equation $\big($see (5.27)$\big)$ takes the explicit form :

$$\varphi_\lambda(x) = \sqrt{\frac{2}{\pi}}\Big\{e^{x\sqrt{\lambda}}\,\frac{1}{\sqrt{\lambda}}\,1_{x \leq 0} + \Big(x + \frac{1}{\sqrt{\lambda}}\Big)1_{x > 0}\Big\} \tag{6.37}$$

Under Q^h, we obtain :

$$X_t = B_t + \int_0^t \frac{du}{X_u^+ + \frac{1}{\sqrt{\lambda}}} \qquad \big(\text{compare with (6.36)}\big) \tag{6.38}$$

where $(B_t,\ t \geq 0)$ is a $\big((\mathcal{F}_t,\ t \geq 0),\ Q^h\big)$ Brownian motion. The martingale $(M_s^h,\ s \geq 0)$ is equal to :

$$M_s^h = \varphi_\lambda(X_s)\exp\Big(-\frac{\lambda}{2}\int_0^s 1_{]-\infty,0]}(X_u)\,du\Big) \tag{6.39}$$

This example motivated us to raise the question : which are the functions h such that the Q^h process is Markovian ? The answer is given by the following :

Proposition B.9. : *Let h be regular, bounded, satisfying equation (6.10) and such that the process $(X_t,\ t \geq 0)$ is Markov under Q^h. Then, there exists $\lambda \geq 0$ such that $h(x) = e^{-\frac{\lambda}{2}x}$ $(x \geq\ 0)$.*

Proof of Proposition B.9. :
To answer this question, we come back to equation (6.32). The problem is to find under which conditions the drift term :

$$\frac{\sqrt{2\pi}\, h(A_t^-)\, 1_{X_t>0} - \Big(\displaystyle\int_0^\infty \frac{dw}{w^{3/2}}\, e^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w)\Big)\, X_t^-}{\sqrt{2\pi}\, h(A_t^-)\, X_t^+ + \displaystyle\int_0^\infty \frac{dw}{w^{1/2}}\, e^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w)} \tag{6.40}$$

does not depend on A_t^-. Considering this expression when $X_t < 0$, the problem amounts to study the functions h for which :

$$\frac{x \displaystyle\int_0^\infty \frac{dw}{w^{3/2}}\, e^{-\frac{x^2}{2w}} h(a+w)}{\displaystyle\int_0^\infty \frac{dw}{w^{1/2}}\, e^{-\frac{x^2}{2w}} h(a+w)} := \psi(x) \tag{6.41}$$

does not depend on a. (6.41) may be written : $\dfrac{\partial}{\partial x} \log\big(\theta(x,a)\big) = -\psi(x)$ where we have denoted : $\theta(x,a) := \displaystyle\int_0^\infty \frac{dw}{\sqrt{w}}\, e^{-\frac{x^2}{2w}} h(a+w)$.
Hence, by integration we obtain the existence of two functions φ_1 and φ_2 such that :

$$\int_0^\infty \frac{x}{\sqrt{2\pi w^3}}\, e^{-\frac{x^2}{2w}} h(a+w)\, dw = \varphi_1(a)\varphi_2(x) \tag{6.42}$$

Letting $x \longrightarrow 0$ in (6.42), we obtain $h(a) = \varphi_1(a)\varphi_2(0)$. Note that the LHS in (6.42) writes $E\big(h(a+T_x)\big)$, where $(T_x,\ x \geq 0)$ is the $\dfrac{1}{2}$-stable subordinator of Brownian first hitting times. Hence we have :

$$E\big[h(a+T_x)\big] = P_x(h)(a) = E\big[\varphi_1(a+T_x)\varphi_2(0)\big] = \varphi_1(a)\varphi_2(0) \tag{6.43}$$

where $(P_x, x \geq 0)$ denotes the semi-group associated with the subordinator $(T_x,\ x \geq 0)$, whose infinitesimal generator is $\Big(\dfrac{\partial^2}{\partial x^2}\Big)^{\frac{1}{2}}$. In other terms, from (6.42), we get :

$$P_x\varphi_1(a) = \frac{\varphi_2(x)}{\varphi_2(0)}\, \varphi_1(a) \tag{6.44}$$

φ_1 is an eigenfunction of P_x, and consequently an eigenfunction of $\dfrac{\partial^2}{\partial x^2}$ $\cdot$ φ_1 being positive and bounded : $\varphi_1(a) = c\, e^{-\frac{\lambda}{2}a}$ $(a, \lambda \geq 0)$ and $h(a) = c\, e^{-\frac{\lambda}{2}a}\varphi_2(0) = c\, e^{-\frac{\lambda}{2}a}$.

7 Part B.3. A local limit theorem for a class of additive functionals of the "long Brownian bridges".

In this section, our aim is to obtain results similar to those in Section B.1, but, now, Brownian motion $(X_s,\ s \geq 0)$ is being replaced by the Brownian bridge with length t, with $t \to \infty$. q denotes a function from $\mathbb{R}$ to $\mathbb{R}_+$, which is Borel, and such that :

$$0 < \int_{-\infty}^{\infty} (1+x^2)\, q(x)\, dx < \infty. \tag{7.1}$$

We let :

$$A_t^q := \int_0^t q(X_s)\,ds \tag{7.2}$$

Theorem B.10.
1) For every x and $y \in \mathbb{R}$, and $\mu > 0$:

$$E_x\Big(\exp\Big(-\frac{\mu}{2}\int_0^t q(X_s)\,ds\Big)\Big|X_t = y\Big) \underset{t\to\infty}{\sim} \frac{\pi}{2}\,\frac{\varphi_{\mu q}(x)\varphi_{\mu q}(y)}{t} \tag{7.3}$$

where $\varphi_{\mu q}$ denotes the unique solution of :

$$\varphi'' = (\mu q)\cdot\varphi, \ \lim_{x\to+\infty}\varphi'(x) = -\lim_{x\to-\infty}\varphi'(x) = \sqrt{\frac{2}{\pi}} \tag{7.4}$$

$$2) \lim_{t\to\infty} \quad t\,P_x\left(A_t^q \in dz|X_t = y\right) = \nu_x * \nu_y\,(dz) \tag{7.5}$$

where ν_x and ν_y have been defined in Theorem B.1. The convergence in (7.5) has the same meaning as in Theorem B.1.

Proof of Theorem B.10.

Without loss of generality, we shall assume that $\mu = 1$.

B.3.1. Lemma B.11.

There exists a constant $C > 0$ such that :

$$E_x\Big(\exp\Big(-\frac{1}{2}\int_0^t q(X_s)\,ds\Big)|X_t = y\Big) \le C\,e^{\frac{(x-y)^2}{2t}}\,\frac{\big(1+|x|\big)\big(1+|y|\big)}{1+t} \tag{7.6}$$

Proof of lemma B.11.

1) As an intermediary result, we already show that :

$$E_x\Big(\exp\Big(-\frac{1}{2}\int_0^t q(X_s)\,ds\Big)|X_t = y\Big) \le C\,e^{\frac{(x-y)^2}{2t}}\,\frac{1+|x|}{\sqrt{1+t}} \tag{7.7}$$

for a constant C which does not depend on x, y, t.
To prove (7.7), we condition with respect to $X_{t/2}$, and we get :

$$\begin{aligned} E_x\Big(\exp\Big(-\frac{1}{2}\int_0^t q(X_s)\,ds\Big)|X_t = y\Big)\,e^{-\frac{(x-y)^2}{2t}} &= \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} E_x\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{t/2} = c\Big) \\ & E_c\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{\frac{t}{2}} = y\Big)\cdot e^{-\frac{(x-c)^2}{t}-\frac{(y-c)^2}{t}}dc \end{aligned} \tag{7.8}$$

In (7.8), we majorize $E_c\Big(\exp-\dfrac{1}{2}A_{t/2}^q|X_{t/2} = y\Big)$ by 1, and we get :

$$\begin{aligned} E_x\Big(\exp-\frac{1}{2}A_t^q|X_t = y\Big)\,e^{-\frac{(x-y)^2}{2t}} &\le \frac{1}{\sqrt{\pi t}}\cdot\int_{-\infty}^{\infty} E_x\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{t/2} = c\Big)\,e^{-\frac{(x-c)^2}{2\cdot t/2}}dc \\ &\le E_x\Big(\exp-\frac{1}{2}A_{t/2}^q\Big) \le C\,\frac{1+|x|}{\sqrt{1+t}} \end{aligned}$$

from Lemma 4.3 in [RVY, I]. Thus, we have obtained (7.7).

2) Then, plugging the estimate (7.7) in (7.8), we obtain :

$$\begin{aligned} E_x\Big(\exp\Big(-\frac{1}{2}\,A_t^q\Big)\big|X_t=y\Big) \;&\leq\; e^{\frac{(x-y)^2}{2t}}\,\frac{C\big(1+|y|\big)}{\sqrt{1+t}}\int_{-\infty}^{\infty} E_x\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{t/2}=c\Big)\,\frac{e^{-\frac{(x-c)^2}{2\cdot t/2}}}{\sqrt{2\pi t/2}}\,dc \\ &\leq\; \frac{C\big(1+|x|\big)\,\big(1+|y|\big)}{1+t}\cdot e^{\frac{(x-y)^2}{2t}} \end{aligned}$$

since :

$$E_c\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{t/2}=y\Big)=E_y\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{t/2}=c\Big)$$

and

$$e^{\frac{(x-y)^2}{2t}}\cdot\frac{C\big(1+|y|\big)}{\sqrt{1+t}}\,E_x\Big(\exp-\frac{1}{2}A_{t/2}^q|X_{t/2}=c\Big)\leq e^{\frac{(x-y)^2}{2t}}\,C\,\frac{\big(1+|y|\big)\big(1+|x|\big)}{1+t}$$

by applying once again Lemma 4.3 in [RVY, I].

B.3.2. Lemma B.12.

Let $Z(t,x,y):=E_x\Big(\exp-\frac{1}{2}A_t^q|X_t=y\Big)$. *On the other hand, let* $U(t,x,y)$ *denote the solution of :*

$$\begin{cases} \dfrac{\partial U}{\partial t}\,(t,x,y)-\dfrac{1}{2}\,\dfrac{\partial^2 U}{\partial x^2}\,(t,x,y)+\dfrac{1}{2}\,U(t,x,y)\,q(x)=0 \\ U(0,\bullet,y)=\delta_y \end{cases} \tag{7.9}$$

Then : $\qquad Z(t,x,y)=\sqrt{2\pi t}\,e^{\frac{(x-y)^2}{2t}}\,U(t,x,y)$.

In particular, it follows from Lemma B.11., that :

$$U(t,x,y)\leq C\,\frac{\big(1+|x|\big)\big(1+|y|\big)}{(1+t)^{3/2}} \tag{7.10}$$

Proof of Lemma B.12.

We know that, for every regular function f :

$$Z^f(t,x):=E_x\Big[\exp\Big(-\frac{1}{2}\,A_t^q\Big)\,f(X_t)\Big]$$

is solution of :

$$\frac{\partial Z^f}{\partial t}-\frac{1}{2}\,\frac{\partial^2 Z^f}{\partial x^2}+\frac{1}{2}\,Z^f\cdot q=0,\quad Z^f(0,x)=f(x) \tag{7.11}$$

It suffices, in order to obtain Lemma B.12, to write :

$$Z(t,x,y)=\lim_{\varepsilon\downarrow 0}\frac{E_x\Big[\Big(\exp\Big(-\frac{1}{2}\,A_t^q\Big)\Big)\,f_\varepsilon(X_t)\Big]}{E_x\big(f_\varepsilon(X_t)\big)}$$

where f_ε is a family of functions which converges weakly towards δ_y, and to use (7.11).

B.3.3. We define, for every $\lambda > 0$:

$$A(\lambda, x, y) = \int_0^\infty e^{-\lambda t} U(t, x, y)\, dt \tag{7.12}$$

Since $Z(t, x, y)$ is a decreasing function of t, we deduce the following equivalences from the Tauberian theorem :

$$
\begin{aligned}
&i) \quad Z(t,x,y) \underset{t\to\infty}{\sim} \frac{\pi}{2}\, \frac{\varphi_q(x)\varphi_q(y)}{t} \\
&ii) \quad U(t,x,y) \underset{t\to\infty}{\sim} \frac{\sqrt{\pi}}{2\sqrt{2}}\, \frac{\varphi_q(x)\varphi_q(y)}{t^{3/2}} \\
&iii) \quad \Big|\frac{\partial}{\partial\lambda}\, A(\lambda,x,y)\Big| = -\frac{\partial}{\partial\lambda}\, A(\lambda,x,y) \underset{\lambda\to 0}{\sim} \frac{1}{2\sqrt{2}}\, \frac{1}{\sqrt{\lambda}}\, \varphi_q(x)\varphi_q(y) \qquad (7.13)
\end{aligned}
$$

We shall now show (7.13). We already deduce from Lemmas B.11 and B.12 that :

$$\lim_{\lambda\to 0} A(\lambda,x,y) \;=\; \int_0^\infty U(t,x,y)\, dt < \infty \tag{7.14}$$

$$A(\lambda,x,y) \;\le\; C\,\big(1+|x|\big)\big(1+|y|\big) \tag{7.15}$$

$$\Big|\frac{\partial}{\partial\lambda} A(\lambda,x,y)\Big| \;\le\; \frac{C}{\sqrt{\lambda}}\,\big(1+|x|\big)\big(1+|y|\big) \tag{7.16}$$

To prove (7.13) we shall show that : $\psi(x,y) := \lim\limits_{\lambda\to 0} \sqrt{\lambda}\, \frac{\partial}{\partial\lambda}\, A(\lambda,x,y)$ satisfies the Sturm-Liouville equation (for any fixed y) :

$$\frac{\partial^2}{\partial x^2}\, \psi = \psi\, q, \quad \text{with adequate limit conditions in } x = \pm\infty \tag{7.17}$$

B.3.3.1. We get, from (7.9) :

$$U(t,x,y) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y)^2}{2t}} - \frac{1}{2}\int_0^t ds \int_{-\infty}^\infty \frac{e^{-\frac{(x-z)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\, U(s,z,y)\, q(z) dz \tag{7.18}$$

Thus, after taking the Laplace transform of the two sides of (7.18), we obtain :

$$A(\lambda,x,y) = g_\lambda(x,y) - \frac{1}{2}\int_{-\infty}^\infty g_\lambda(x,z)\, A(\lambda,z,y)\, q(z) dz \tag{7.19}$$

where g_λ denotes the density of the resolvent kernel of Brownian motion :

$$g_\lambda(x,z) = \frac{1}{\sqrt{2\lambda}}\, e^{-|x-z|\sqrt{2\lambda}}$$

We write (7.19) in the form :

$$A(\lambda,x,y) = G_\lambda\Big[\delta_y - \frac{1}{2}\,\big(A(\lambda,\bullet,y)\, q(\bullet)\Big](x) \tag{7.20}$$

with for any Radon measure $\mu(dz)$:

$$G_\lambda \mu\,(x) := \int_{-\infty}^\infty g_\lambda(x,z)\, \mu(dz) \tag{7.21}$$

and we use the resolvent equation : $\dfrac{\partial^2}{\partial x^2} G_\lambda \mu = -2\mu + 2\lambda G_\lambda \mu$, to obtain :

$$\frac{\partial^2}{\partial x^2} A(\lambda,x,y) = 2\lambda\, A(\lambda,x,y) - \big[2\delta_y - A(\lambda,x,y)\, q(x)\big] \tag{7.22}$$

As a consequence, differentiating with respect to λ, then multiplying by $\sqrt{\lambda}$, we obtain :

$$\frac{\partial^2}{\partial x^2}\Big(\sqrt{\lambda}\,\frac{\partial A}{\partial \lambda}(\lambda,x,y)\Big) - \sqrt{\lambda}\,\frac{\partial A}{\partial \lambda}(\lambda,x,y)\, q(x) = 2\sqrt{\lambda}\, A(\lambda,x,y) + 2\lambda^{3/2}\frac{\partial A}{\partial \lambda}(\lambda,x,y) \tag{7.23}$$

Hence, from (7.16) and (7.15), and denoting $\widetilde{A}(\lambda,x,y) := \sqrt{\lambda}\,\dfrac{\partial A}{\partial \lambda}(\lambda,x,y)$, it follows that :

$$\Big|\frac{\partial^2}{\partial x^2}\big(\widetilde{A}(\lambda,x,y)\big) - \widetilde{A}(\lambda,x,y)\, q(x)\Big| \le C\,\sqrt{\lambda}\big(1+|x|\big)\big(1+|y|\big) \qquad (\lambda \to 0) \tag{7.24}$$

(7.24) is the first step to prove that $\widetilde{A}(\lambda,x,y)$ converges, as $\lambda \to 0$, to a solution of the Sturm-Liouville equation (7.17).

B.3.3.2. We now examine the limit conditions in $x = \pm\infty$.

We come back to equation (7.19) which we differentiate with respect to λ, then we multiply by λ :

$$\begin{aligned}\sqrt{\lambda}\widetilde{A}(\lambda,x,y) \;&=\; -\frac{1}{2}A(\lambda,x,y) - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|\sqrt{2\lambda}}|x-z|\,\big(\delta_y(dz) - A(\lambda,z,y)\,q(z)dz\big)\\ &\quad -\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-|x-z|\,\sqrt{2\lambda}}\widetilde{A}(\lambda,x,z)\,q(z)dz\end{aligned} \tag{7.25}$$

From (7.16) and (7.14), respectively we deduce that :

$$\sqrt{\lambda}\,\widetilde{A}(\lambda,x,y) \underset{\lambda\to 0}{\longrightarrow} 0 \qquad \text{and} \quad A(\lambda,x,y) \quad \text{converges as} \quad \lambda \to 0.$$

hence, from (7.25), since $\displaystyle\int_{-\infty}^{\infty}(1+x^2)\,q\,(dz) < \infty$:

$$\lim_{\lambda\to 0}\int_{-\infty}^{\infty} e^{-|x-z|\,\sqrt{2\lambda}}\widetilde{A}(\lambda,x,z)\,q(z)dz \quad \text{exists.} \tag{7.26}$$

On the other hand, differentiating (7.25) with respect to x, we obtain :

$$\frac{\partial \widetilde{A}}{\partial x}(\lambda,x,y) = \frac{\partial B}{\partial x} - \frac{1}{2}\Big\{\int_{-\infty}^{x} e^{-|x-z|\,\sqrt{2\lambda}}\widetilde{A}(\lambda,z,y)\,q(z)dz + \int_{x}^{\infty} e^{-|x-z|\,\sqrt{2\lambda}}\widetilde{A}(\lambda,z,y)\,q(z)dz\Big\} \tag{7.27}$$

with

$$B := -\frac{1}{2\sqrt{\lambda}}\Big\{A + \int_{-\infty}^{\infty} e^{-|x-z|\,\sqrt{2\lambda}}\big(\widetilde{A}(\lambda,z,y)\,q(z)dz - \delta_y(dz)\big)\tag{7.28}$$

We deduce from (7.27), (7.26) and (7.28) that :

$$\lim_{\substack{\lambda\to 0\\ x\to\infty}} \frac{\partial}{\partial x}\,\widetilde{A}(\lambda,x,y) = -\lim_{\substack{\lambda\to 0\\ x\to -\infty}} \frac{\partial}{\partial x}\,\widetilde{A}(\lambda,x,y) = C(y) \tag{7.29}$$

$\big($cf [RVY, I], p. 194-197 for similar computations$\big)$. Thus, from the equivalence between i), ii) and iii) which we recalled in (7.13), we get :

$$E_x\Big[\exp - \frac{1}{2}\int_0^t q(X_s)ds\big|X_t = y\Big] \underset{t\to\infty}{\sim} \frac{\psi(x,y)}{t} \tag{7.30}$$

where ψ is solution to :

$$\frac{\partial^2\psi}{\partial x^2}(x,y) = \psi(x,y)\,q(x), \ \lim_{x\to+\infty}\frac{\partial\psi}{\partial x}(x,y) = -\lim_{x\to-\infty}\frac{\partial\psi}{\partial x}(x,y) = C(y) \tag{7.31}$$

Thus, from the definition of φ_q $\big($see(7.4)$\big)$, we get :

$$\psi(x,y) = C(y)\sqrt{\frac{\pi}{2}}\,\varphi_q(x)$$

Now, since $Z(t,x,y)$ is symmetric in x and y :

$$\psi(x,y) = K\varphi_q(x)\varphi_q(y) \tag{7.32}$$

It remains to determine the value of K. For this purpose, we write :

$$\varphi_q(x) = E_x\Big(\Big(\exp\Big(-\frac{1}{2}\int_0^t ds\,q(X_s)\Big)\Big)\,\varphi_q(X_t)\Big)$$

$\Big($since $\varphi_q(X_t)\exp\Big(-\frac{1}{2}A_t^q\Big)$, $t\geq 0$ is a martingale$\Big)$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} E_x\big(\exp\big(-\frac{1}{2}A_t^q\big)\big|X_t = y\big)\,\frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}\,\varphi_q(y)dy \\
&\underset{t\to\infty}{\sim} K\int_{-\infty}^{\infty}\frac{\varphi_q(x)\varphi_q(y)}{t}\,\varphi_q(y)\,\frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}\,dy \\
&= \frac{K\ \varphi_q(x)}{t}\,E_x\big(\varphi_q^2(X_t)\big) \underset{t\to\infty}{\sim} \frac{K\ \varphi_q(x)}{t}\,\frac{2}{\pi}\,t
\end{aligned}$$

since $\varphi_q(z) \underset{|z|\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\,|z|$. Hence $K\frac{2}{\pi} = 1$ and $K = \frac{\pi}{2}\cdot$

Thus, we record our result, which is point 1 of Theorem B.10.

4) Point 2 of Theorem B.10 may be proven, which the help of (7.3), exactly as Theorem B1.1.

Remark B.13. Under our hypothesis H1 on q, there is the equivalence :

$$Z(t,x,y) \equiv E_x\Big(\exp - \frac{1}{2}\int_0^t q(X_s)ds\big|X_t = y\Big) \underset{t\to\infty}{\sim} \frac{\pi}{2t}\,\varphi_q(x)\varphi_q(y)$$

Intuitively, we may think of the bridge of duration t going from x to y as "resembling", as $t\to\infty$, to the concatenation of two brownian motions each being defined on a time interval $\Big[0,\frac{t}{2}\Big]$, with the first one starting from x to y as "resembling", as $t\to\infty$, to the concatenation of two brownian motions each being defined on a time interval $\Big[0, \dfrac{t}{2}\Big]$, with the first one starting from x and the second one, after time reversal, starting from y, these two parts being independent. If this were true, then :

$$\begin{aligned} Z(t,x,y) &= E_x\Big(\exp - \frac{1}{2}\, A_t^q | X_t = y\Big) = E_x\Big(\exp - \frac{1}{2}\, A_{t/2}^q\Big) \cdot E_y\Big(\exp - \frac{1}{2}\, A_{t/2}^q\Big) \\ &\underset{t\to\infty}{\sim} \frac{\varphi_q(x)}{\sqrt{t/2}} \cdot \frac{\varphi_q(y)}{\sqrt{t/2}} = \frac{4}{\pi}\,\Big(\frac{\pi}{2}\,\frac{\varphi_q(x)\varphi_q(y)}{t}\Big) \end{aligned}$$

Note that the factor $\dfrac{4}{\pi}$ which we just obtained measures, in some sense, the default of independence of these two brownian components.

Remark B.14. Theorem B.10. allows to "penalize long Brownian Bridges". More precisely, for every $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\frac{E_x\Big(1_{\Lambda_s} \exp\Big(-\frac{1}{2}\, A_t^q\Big)|X_t = y\Big)}{E_x\Big(\exp\Big(-\frac{1}{2}\, A_t^q\Big)|X_t = y\Big)} \underset{t\to\infty}{\longrightarrow} E_x(1_{\Lambda_s} M_s^{\varphi}) \tag{7.33}$$

with $M_s^{\varphi} := \dfrac{\varphi_q(X_s)}{\varphi_q(x)} \exp\Big(-\dfrac{1}{2}\, A_s^q\Big)$, et $(M_s^{\varphi},\ s \geq 0)$ is a positive martingale. In other terms, comparing with Theorem 5.1 in [RVY, I], the penalisation is the same for "long bridges" as for Brownian motion itself. Once more (see [RVY, III]), we obtain that a long bridge of duration t, as $t \to \infty$, behaves as a free Brownian motion.

Finally, we show (7.33).

$$\begin{aligned} \frac{E_x\Big(1_{\Lambda_s} \exp - \frac{1}{2}\, A_t^q | X_t = y\Big)}{E_x\Big(\exp - \frac{1}{2}\, A_t^q | X_t = y\Big)} &= \frac{E_x\Big(1_{\Lambda_s}\Big(\exp - \frac{1}{2} A_s^q\Big)\, E_{X_s,s}\Big(\exp - \frac{1}{2}\displaystyle\int_s^t q(X_u)du | X_t = y\Big)\Big)}{E_x\Big(\exp - \frac{1}{2}\, A_t^q | X_t = y\Big)} \\ &\underset{t\to\infty}{\sim} \frac{E_x\Big(1_{\Lambda_s} \exp\Big(-\frac{1}{2}\, A_s^q\Big) \cdot \dfrac{\varphi_q(X_s)\varphi_q(y)}{t-s}\Big)}{\frac{\varphi_q(x)\varphi_q(y)}{t}} \underset{t\to\infty}{\longrightarrow} E_x(1_{\Lambda_s} M_s^{\varphi}). \end{aligned}$$

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